Canonical Domain Reduction for Partial Counterfactual Identification

Yesong Choe · Yeahoon Kwon · Min Woo Park · Sanghack Lee | Causality Lab., GSDS, SNU

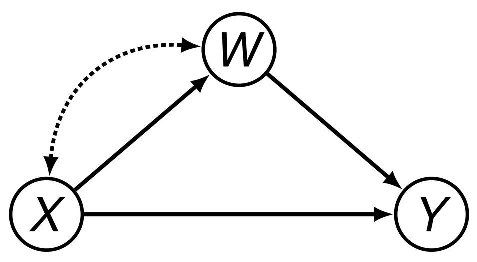
Seoul National University
Graduate School of Data Science

1. Partial identification: fix $\mathcal{Q}$ and $\mathcal{I}$

Partial-identification task. For a fixed target $\mathcal{Q}$ and available information $\mathcal{I}$, return the sharp interval compatible with $\mathcal{I}$:

$$\boxed{\left[\underline{\mathcal{Q}}(\mathcal{I}), \overline{\mathcal{Q}}(\mathcal{I})\right]}.$$

Running example: a binary nested counterfactual



Binary NDE graph
dashed arc: latent X–W dependence

Nested intervention

- **Direct path:** send 0 along $X \to Y$.
- **Mediator path:** generate W_1, then pass it to Y.
- **Outcome:** record whether $Y = 1$.

$$\mathcal{Q} = P(Y_{0,W_1} = 1).$$

2. One exact baseline: a 128-state canonical LP

One exact construction enumerates deterministic mechanisms.
One deterministic mechanism = seven binary response entries

X	W_0	W_1	Y_{00}	Y_{01}	Y_{10}	Y_{11}
natural	W response		Y response			

$$u = (X; W_0, W_1; Y_{00}, Y_{01}, Y_{10}, Y_{11}), \qquad |\Omega_{\text{full}}| = 2^1\, 2^2\, 2^4 = 128.$$

The reference LP assigns mass ψ_u to each mechanism and optimizes:

$$\underline{\mathcal{Q}}, \overline{\mathcal{Q}} = \min / \max_{\psi \geq 0}\ c^\top \psi,$$
$$\text{s.t.} \quad M\psi = p, \qquad \mathbf{1}^\top \psi = 1.$$

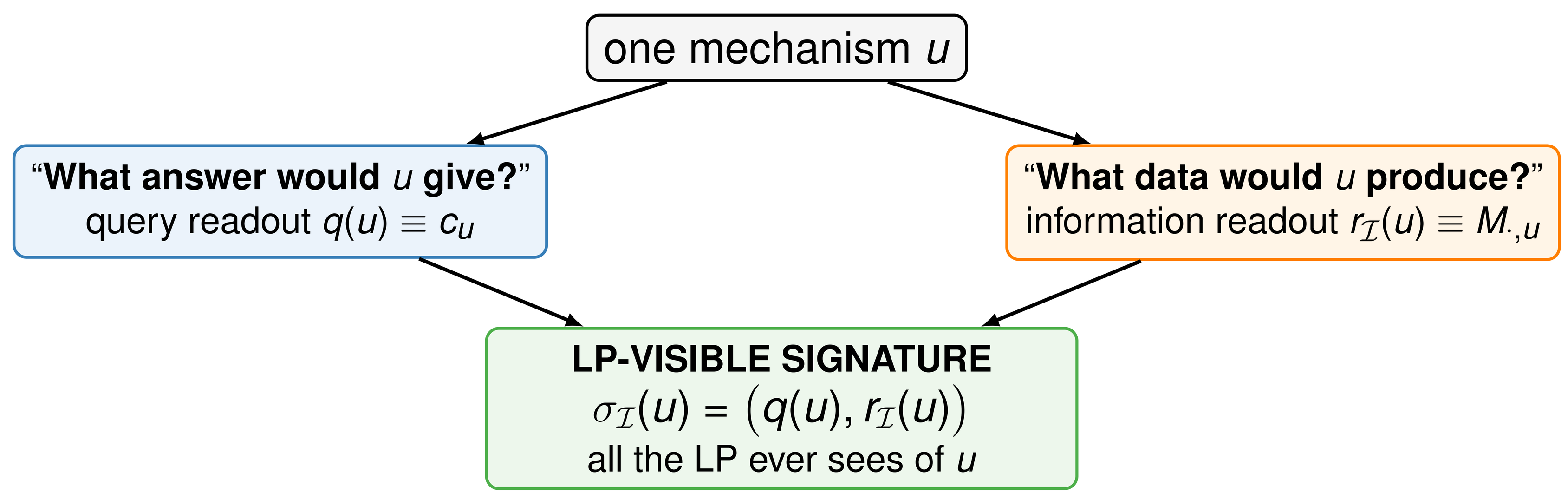
- **Bottleneck:** the catalogue of complete mechanisms grows exponentially.

Exact baseline, not a fixed-problem minimum. The catalogue stores complete mechanisms. Its size is not tailored to this fixed $(\mathcal{Q}, \mathcal{I})$, so the LP may not read every distinction it records.

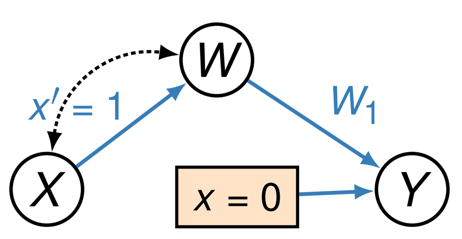
3. LP-visible signature: two readouts

Research question. For fixed $(\mathcal{Q}, \mathcal{I})$, which states of this reference LP can be pooled without changing its sharp bounds?

The fixed $(\mathcal{Q}, \mathcal{I})$ LP asks only two questions about each mechanism u:



Query readout

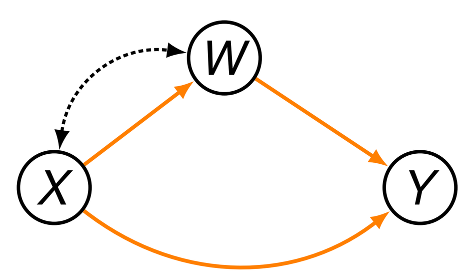


Counterfactual readout for u:

$$q(u) = \mathbb{1}\{Y_{0,W_1}(u) = 1\},$$
$$q(u) \in \{0, 1\}.$$

- Read W_1, then select Y_{00} or Y_{01}.
- Only 3 of 7 entries are used, giving 8 query-level states.

Observational readout



Factual readout for u:

$$(x, w, y) = (X, W_X, Y_{X,W_X})(u),$$
$$r_{\text{obs}}(u) = \mathbf{e}_{(x,w,y)}.$$

- Follow the factual path $X \to W_X \to Y_{X,W_X}$.
- Output one of eight observed cells (x, w, y).

States with the same two readouts are interchangeable to the fixed LP:

Merge rule: $u \sim_{\mathcal{I}} v \iff q(u) = q(v)$ and $r_{\mathcal{I}}(u) = r_{\mathcal{I}}(v)$.

Why the merge is safe. If $u \sim_{\mathcal{I}} v$, moving mass between them leaves both $c^\top\psi$ and $M\psi$ unchanged. Thus the LP only needs the total mass of each signature class.

The quotient acts on the specified baseline LP; it does not add graphical restrictions.

4. Exact quotient: merge identical readouts

Pooling equal-signature mechanisms gives a smaller LP with the same attainable target range as the specified baseline.

Full-observation signature:

$$\sigma_{\text{obs}}(u) = \left[q(u) \mid X(u), W_X(u), Y_{X,W_X}(u)\right].$$

	X	W_0	W_1	Y_{00}	Y_{01}	Y_{10}	Y_{11}	$q(u)$	(x, w, y)
u^a	0	0	0	1	0	0	0	1	(0, 0, 1)
u^b	0	0	0	1	1	1	1	1	(0, 0, 1)
u^c	0	0	1	1	0	0	0	0	(0, 0, 1)
u^d	1	0	0	1	0	0	0	1	(1, 0, 0)

- ✓ u^a, u^b: same query bit, same factual cell ⇒ **merge**.
- u^c: different query bit ⇒ **keep separate**.
- ✗ u^d: $T(u^a) = T(u^d) = (0, 1, 0)$, but their factual cells differ ⇒ a query-only merge is unsafe.

One pooled variable per signature class:

$$\phi_t = \sum_{u:\, \sigma_{\mathcal{I}}(u) = t} \psi_u.$$

Exact—not an approximation

The objective and every constraint rewrite through the pooled ϕ.

$$\phi = \Pi\psi, \qquad c = \Pi^\top \tilde{c}, \qquad M = \widetilde{M}\Pi.$$

$$\Longrightarrow \qquad \left[\underline{\mathcal{Q}}, \overline{\mathcal{Q}}\right]_{\text{full}} = \left[\underline{\mathcal{Q}}, \overline{\mathcal{Q}}\right]_{\text{reduced}}.$$

Grouping by $\sigma_{\mathcal{I}}(u) = (c_u, M_{\cdot,u})$ guarantees this factorization.

5. The information set changes the answer

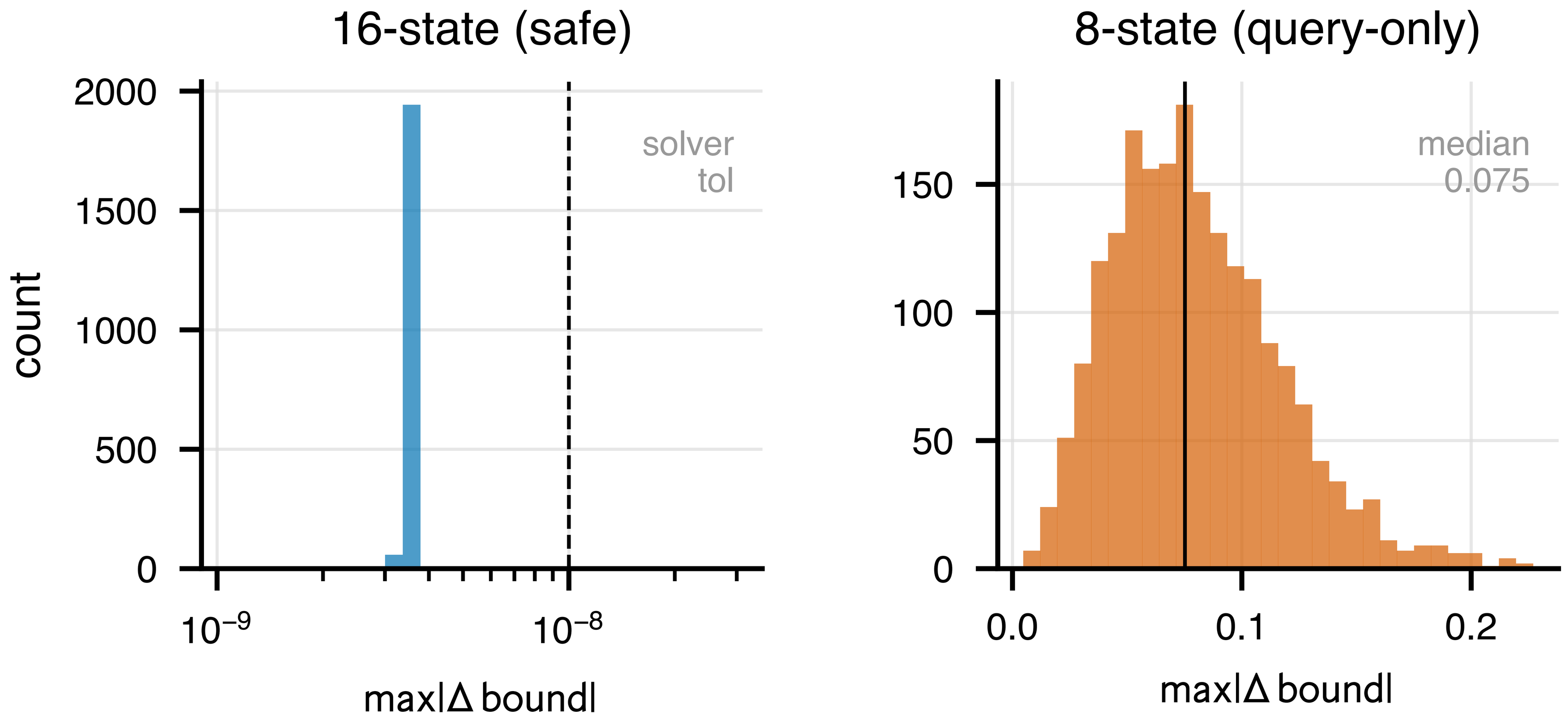
Query-only representation:

$$T(u) = \left(W_1(u), Y_{00}(u), Y_{01}(u)\right) \in \{0, 1\}^3.$$

Eight states describe the query. The information constraints decide whether eight are enough.

Information	Readout retained	Verdict
Three marginals $P(W_1), P(Y_{00}), P(Y_{01})$	$T(u) = (W_1, Y_{00}, Y_{01})$	✓ 8 states **exact**
Full $P_{\text{obs}}(X, W, Y)$	query bit $q(u)$ and factual cell (X, W_X, Y_{X,W_X})	✓ 16 states **exact**
Full P_{obs}, keep T only	factual cell is lost; P_{obs} constraints cannot be recovered	✗ 8 states **unsafe in general**

Full-P_{obs} stress test: 2,000 feasible instances



- ✓ **Safe 16:** all endpoints match the 128-state LP (max $|\blacksquare| \leq 3.7 \times 10^{-9}$).
- ✗ **Query-only 8:** looser on every instance (median 0.075; p99 ≈ 0.19).

- **Larger ternary-mediator example:** 4608 → 48 states; endpoint discrepancy 3.4×10^{-9}; 118.6× median cached ECOS `solve_time` over 20 seeds (not end-to-end runtime).
- **Scope:** finite discrete canonical domains and linear functionals.
- **Limitation:** savings depend on the joint objective–constraint signature; query-context counting alone does not certify arbitrary constraints.

This work was partly supported by the IITP (RS-2022-II220953/50%) and NRF (RS-2023-00211904/50%) grants funded by the Korean government. Yesong Choe was supported in part by Basic Science Research Program through the NRF funded by the Ministry of Education (RS-2025-25418030).