Seoul National University
Graduate School of Data Science

Complete Graphical Criterion for Sequential Covariate Adjustment in Causal Inference

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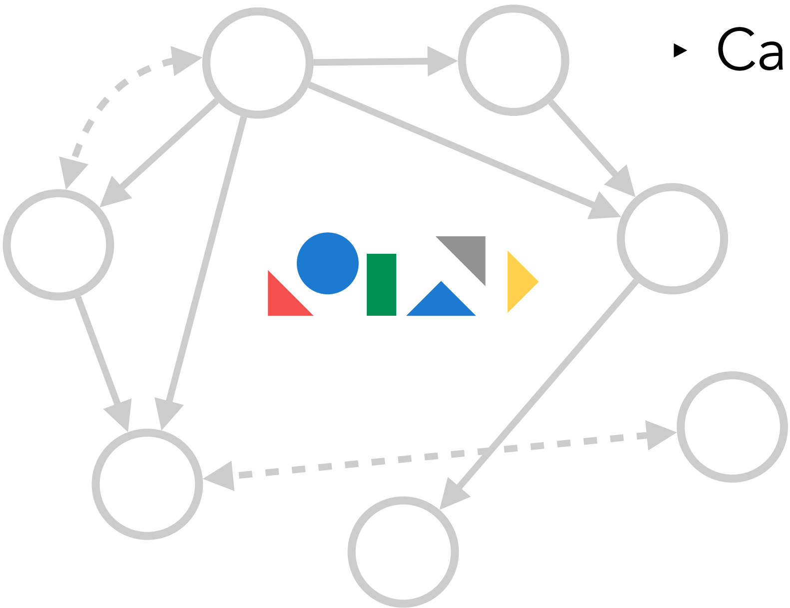
Contents

Causality Lab

Causality Lab

- Causal Bandit
- Causal RL
- Causal Discovery
- Causal Identification
- Causal Estimation



- Causal Representation Learning
- Causal NLP
- Causal Machine Learning
- Causal Recommendation
- Causal Fairness
- Causal Explainability

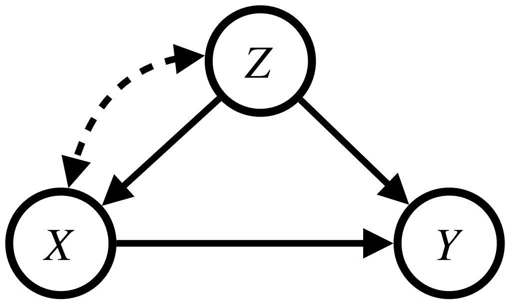
Causal Inference

Causality

Causal Inference

- Identification
- Estimation
- Discovery

Causality



given a *true* causal diagram

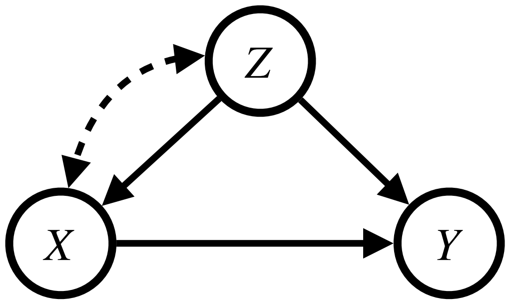
Identification

Estimation

Discovery

Whether the causal effect $P(\mathbf{y} \mid do(\mathbf{x}))$ can be expressed in terms of the observational distribution.

Causality



given a *true* causal diagram

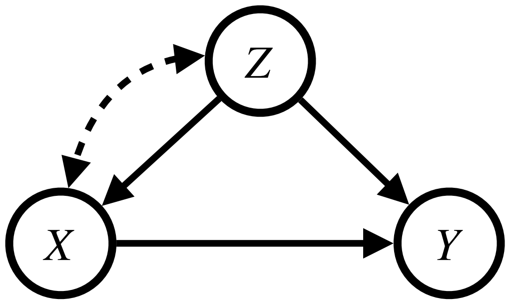
Identification

Estimation

Discovery

Efficient estimation of causal effect $P(\mathbf{y} \mid do(\mathbf{x}))$ from the observational distribution.

Causality



unknow causal diagram

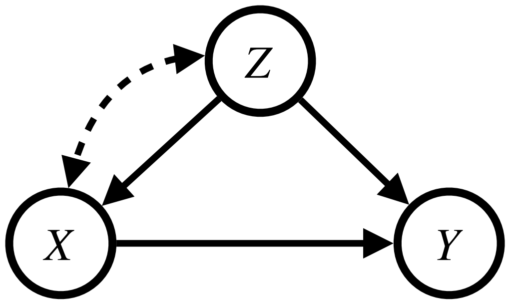
Identification

Estimation

Discovery

Discovery of a causal diagram from an observational datasets.

Causality



given a *true* causal diagram

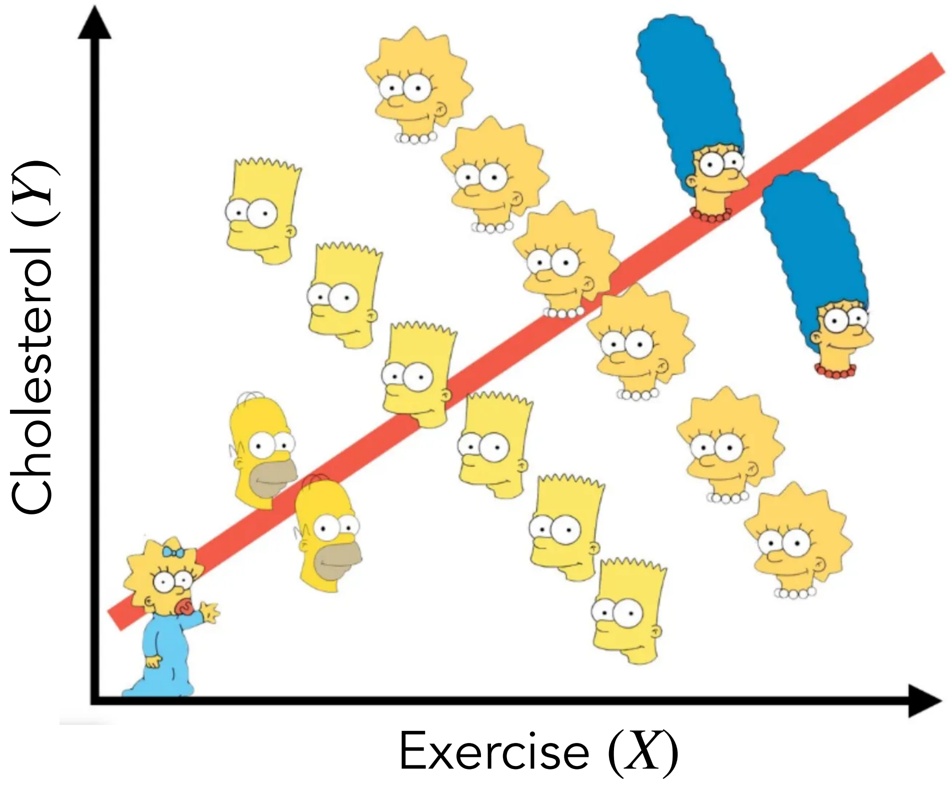
Identification

Estimation

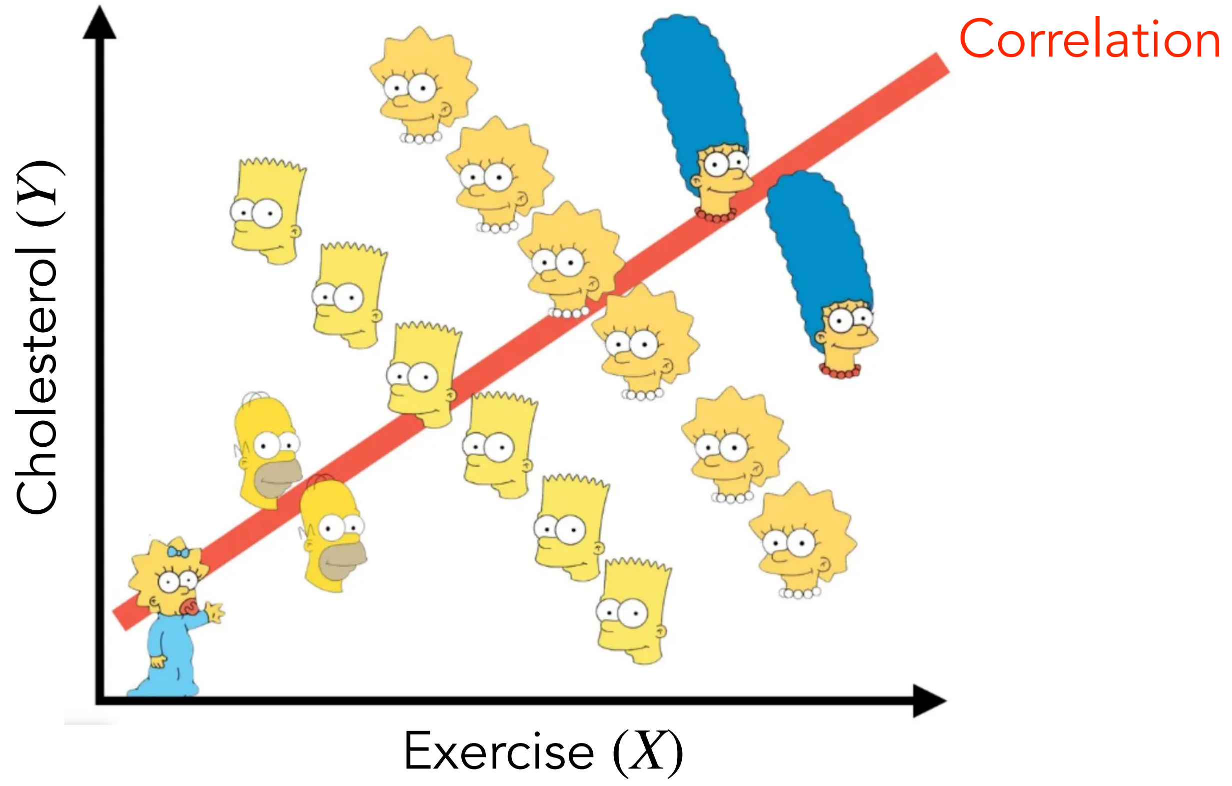
Discovery

Whether the causal effect $P(\mathbf{y} \mid do(\mathbf{x}))$ can be expressed in terms of the observational distribution.

Correlation vs Causation

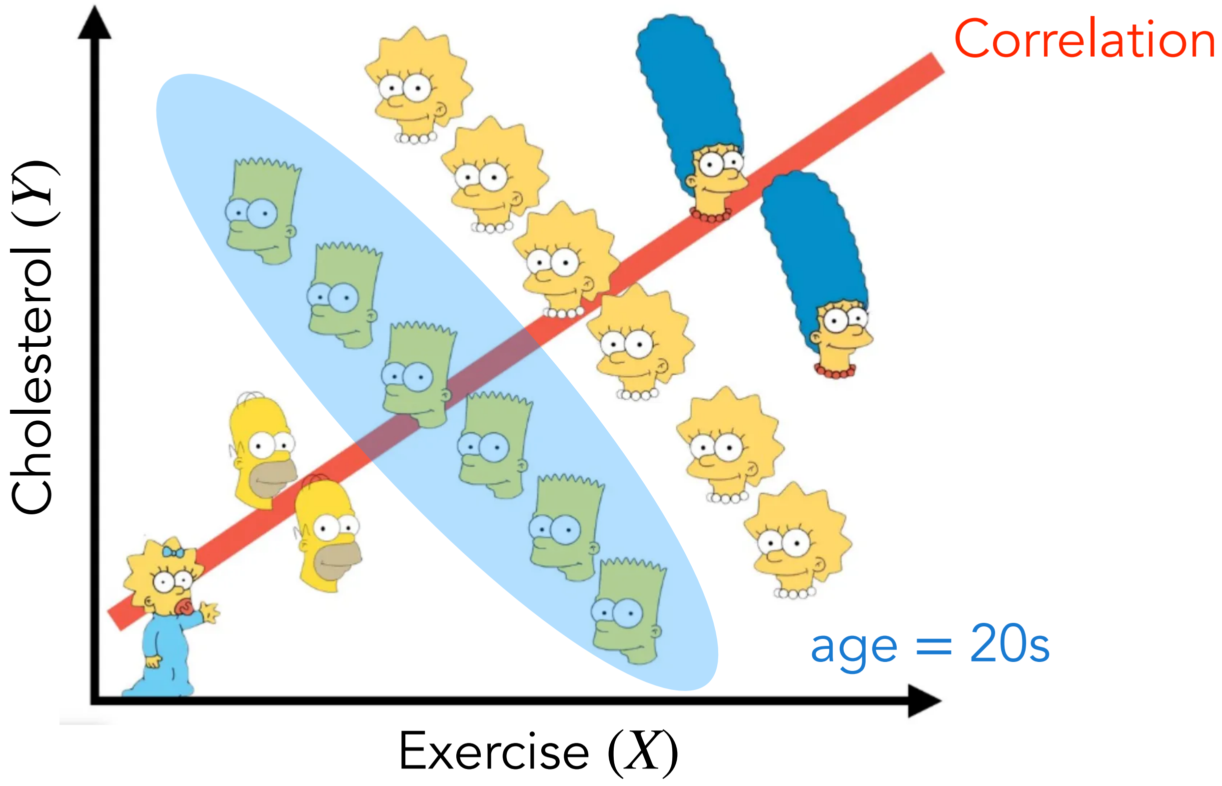


Correlation vs Causation

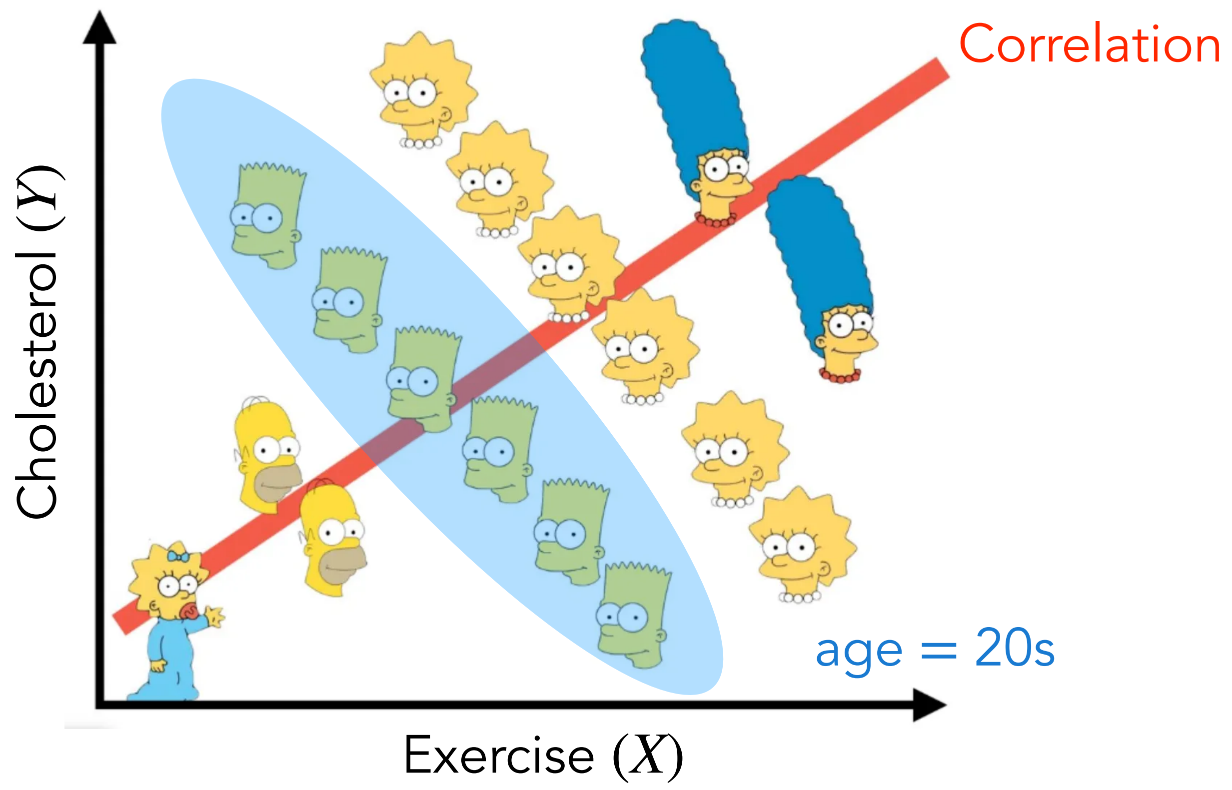


More exercise $\Rightarrow$ More Cholesterol?

Correlation & Causation

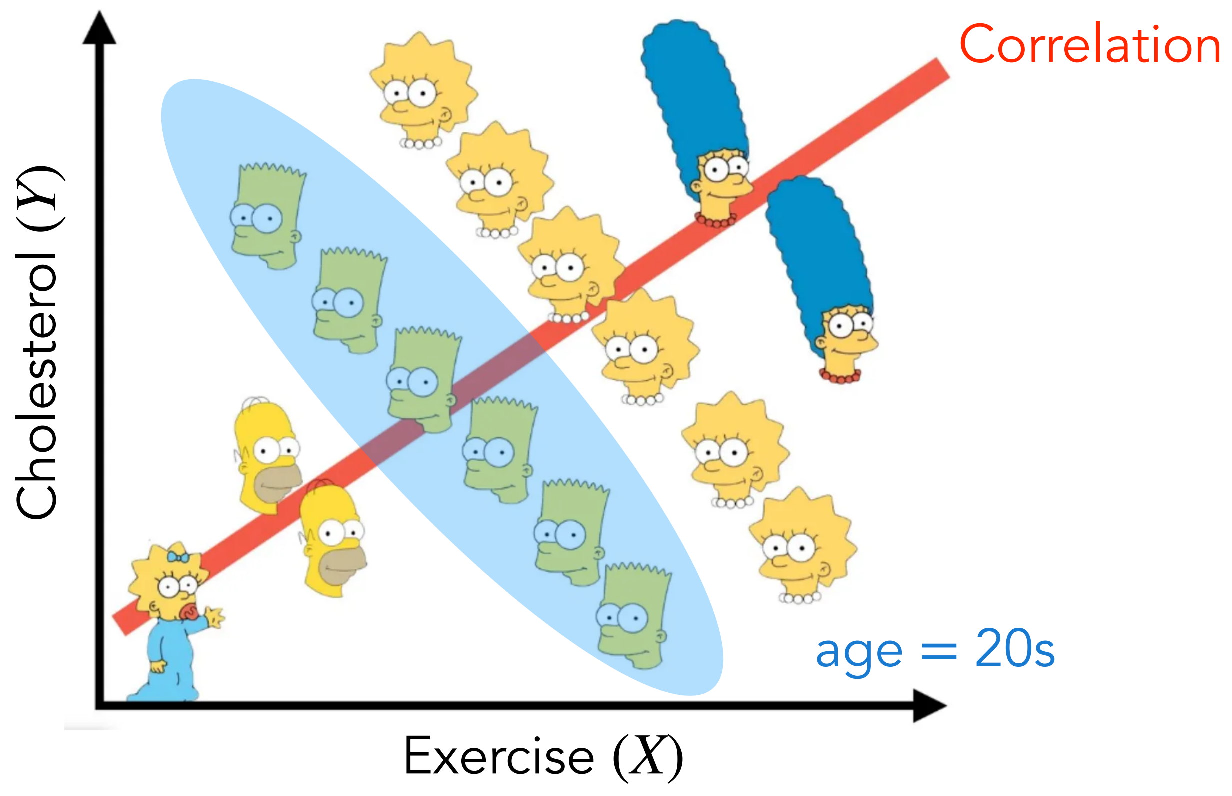


Correlation & Causation



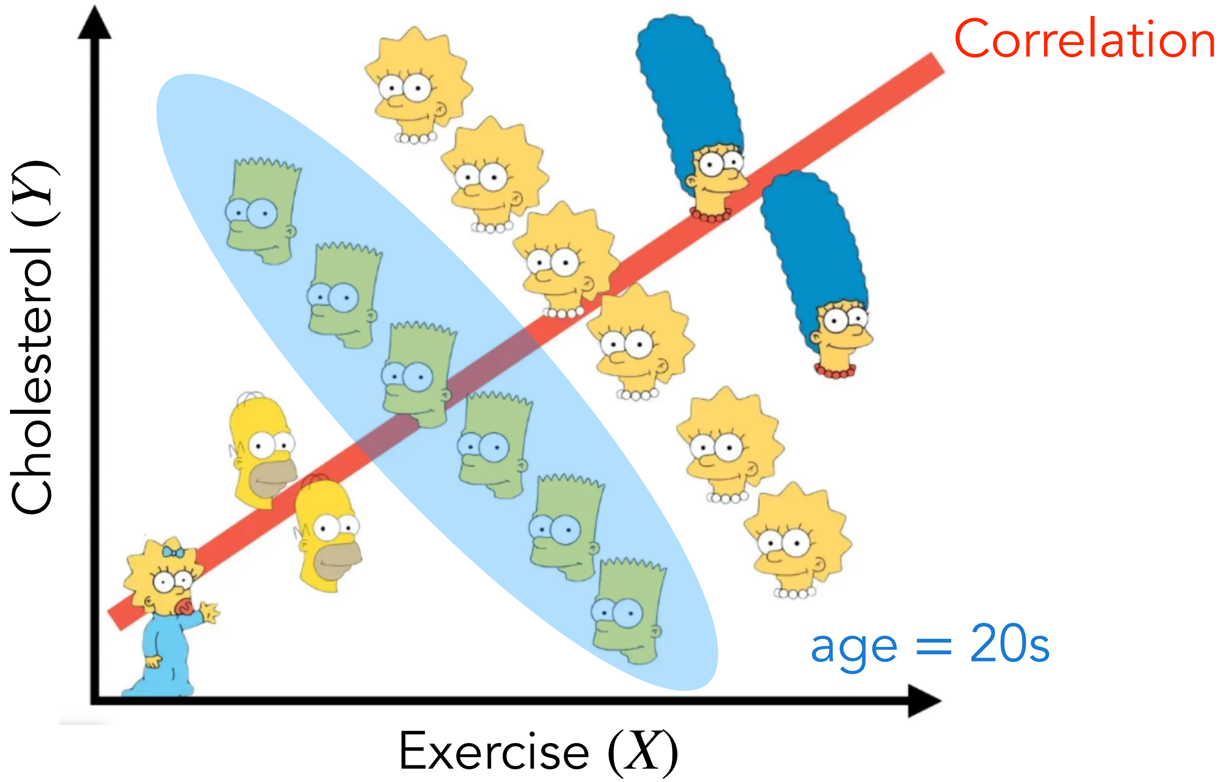
More exercise ⇒ Lower Cholesterol (per age group)

Correlation & Causation



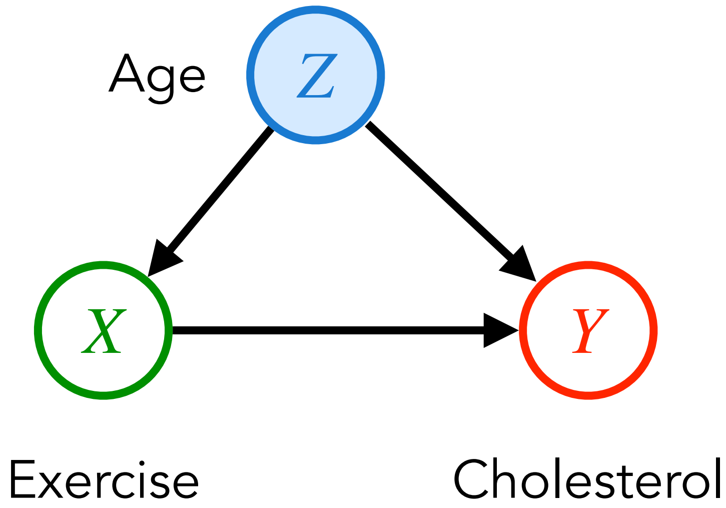
This difference is called confounding bias.

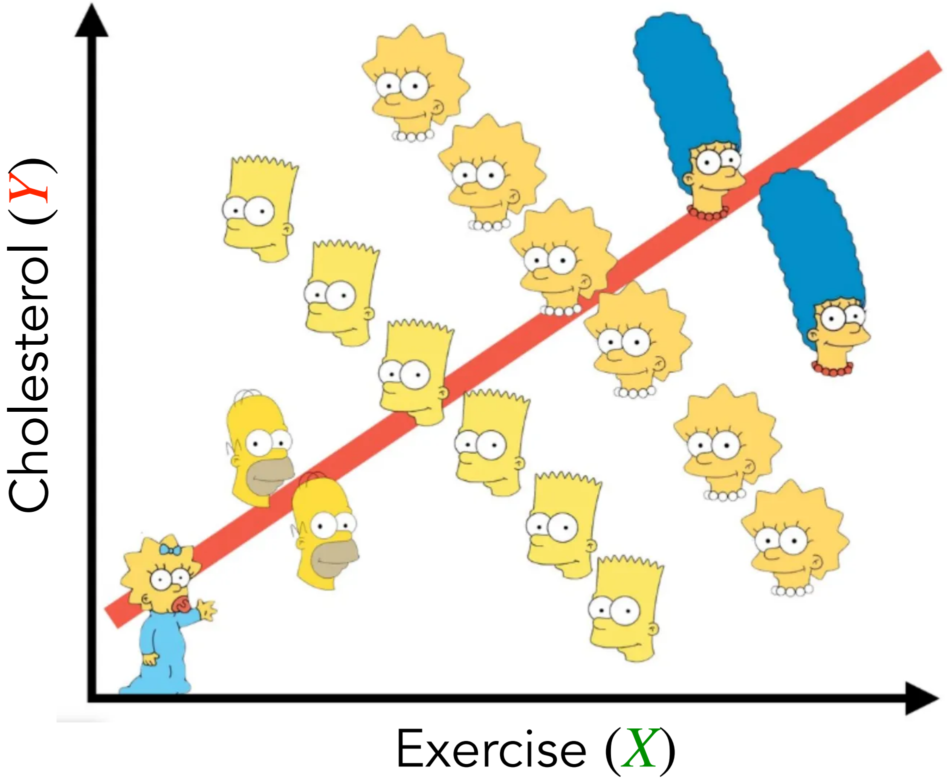
Correlation & Causation



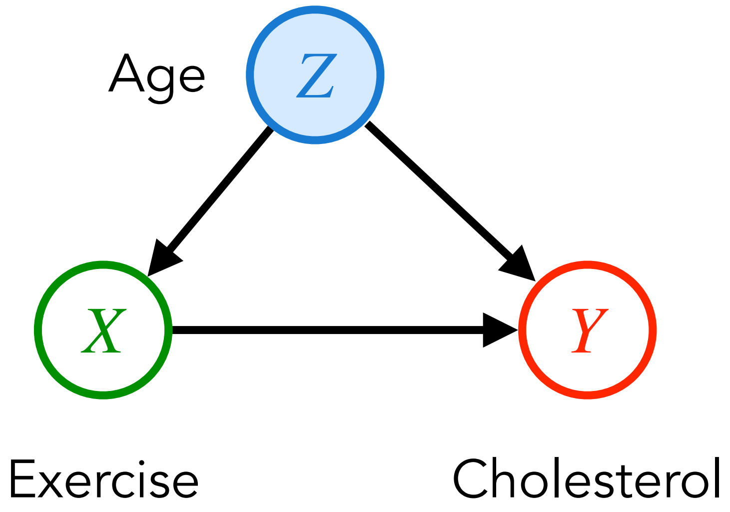
Q. Is confounding bias removable?

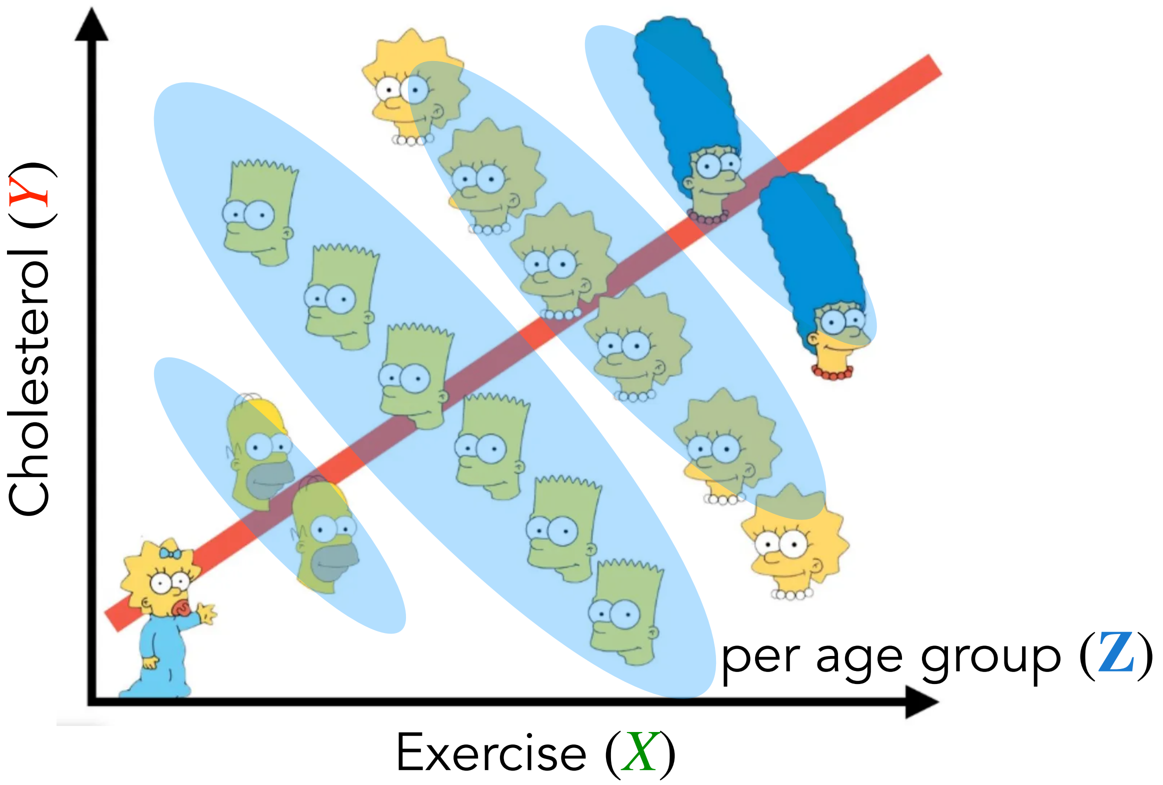
Covariate Adjustment





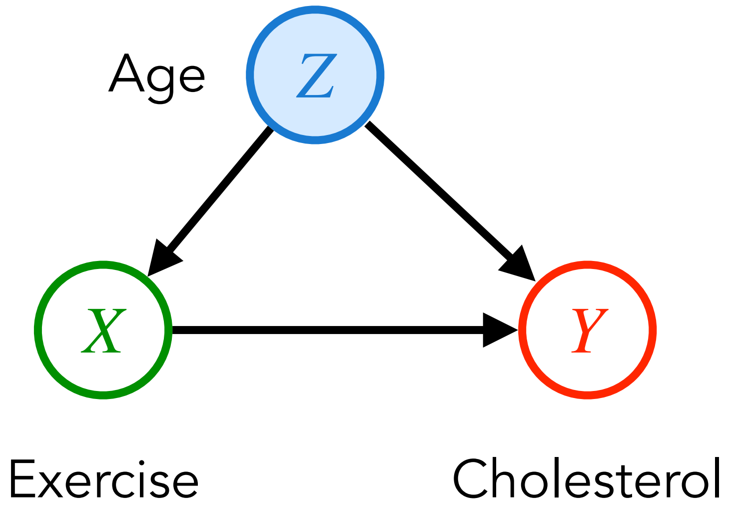
Covariate Adjustment

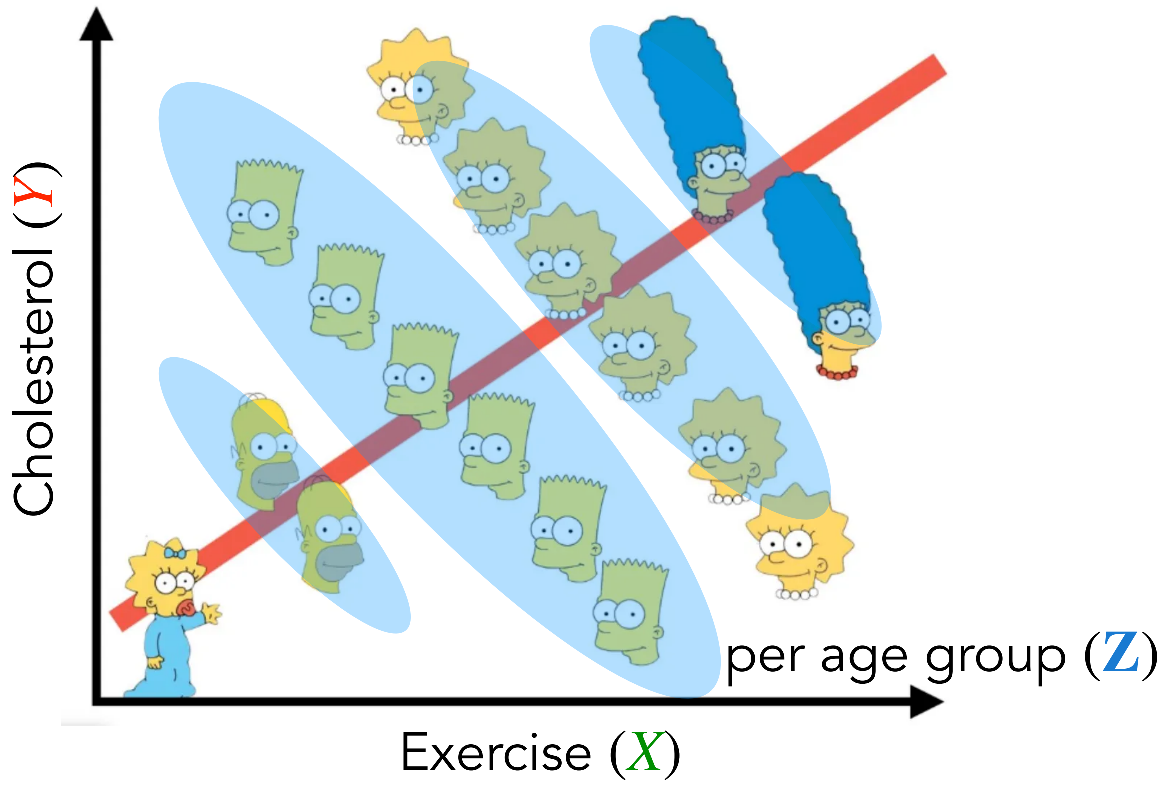




We can compute the causal effect $P(\mathbf{y} \mid do(\mathbf{x}))$ by adjusting the confounder $\mathbf{Z}$.

Covariate Adjustment



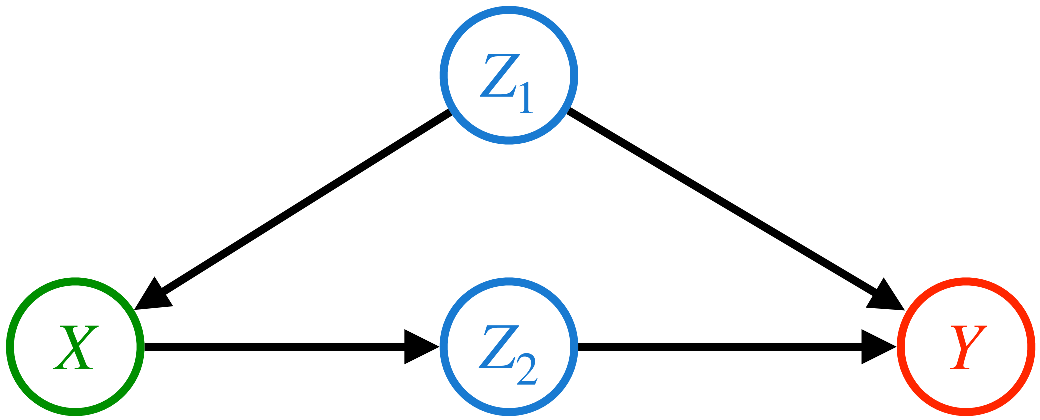


We can compute the causal effect $P(\mathbf{y} \mid do(\mathbf{x}))$, adjusting the confounder $\mathbf{Z}$.

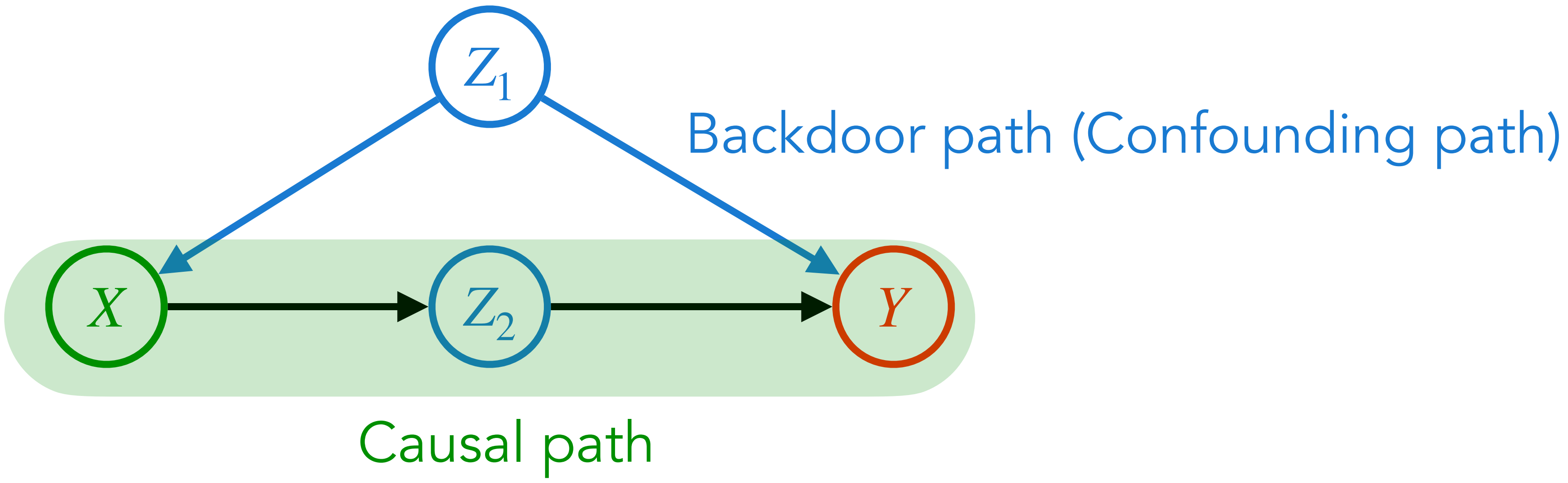
Covariate Adjustment (CA): $P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})$

Backdoor

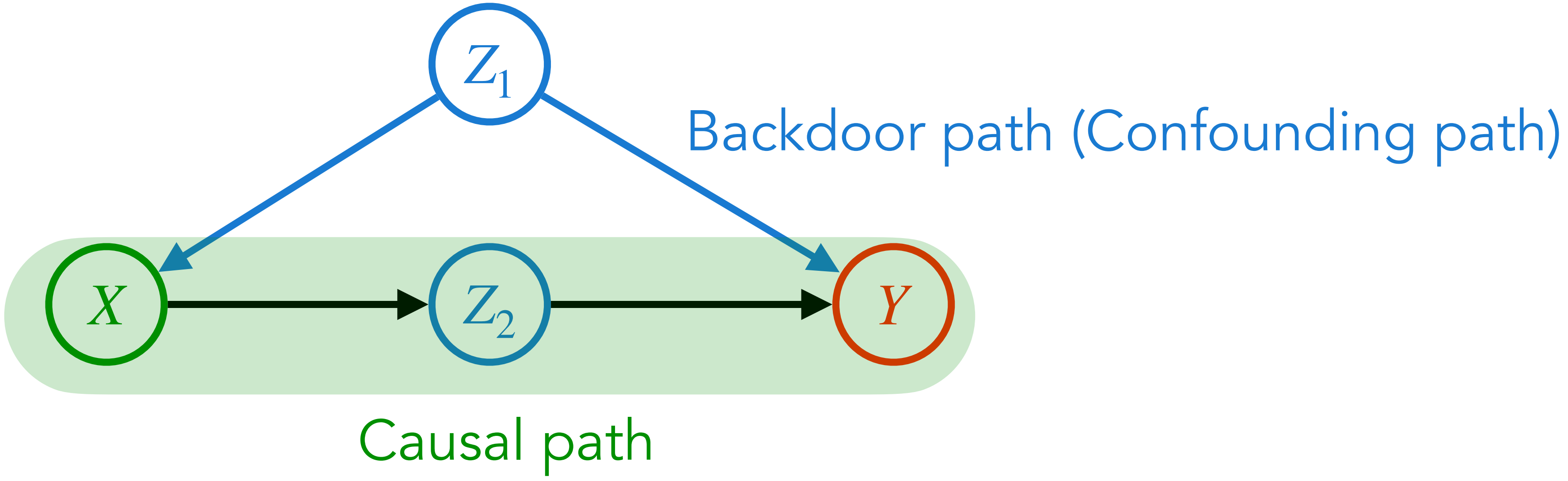
Backdoor path



Backdoor path

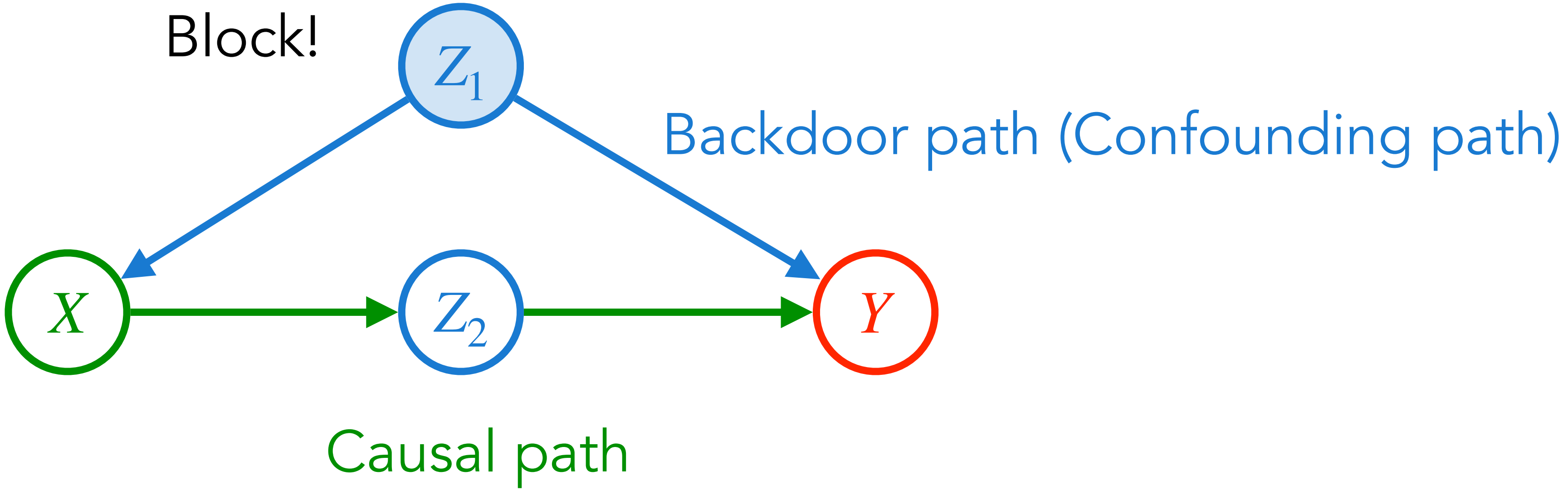


Backdoor path



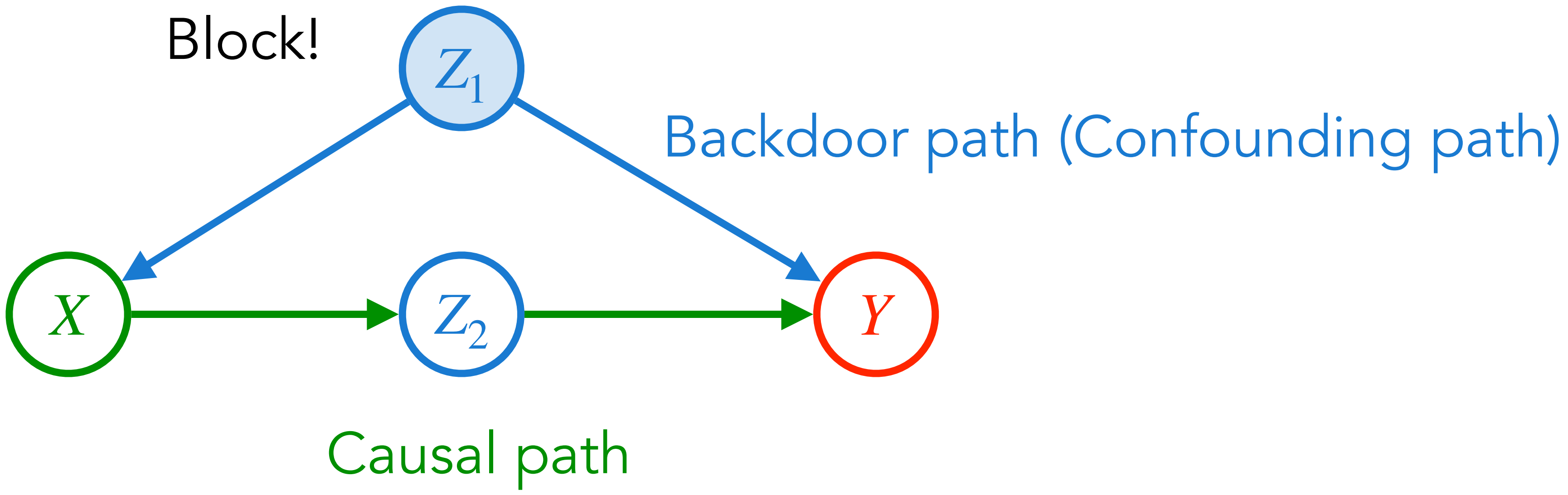
Correlation = Causal effect + Counfounding bias

Backdoor path



Correlation = Causal effect + Counfounding bias

Backdoor path

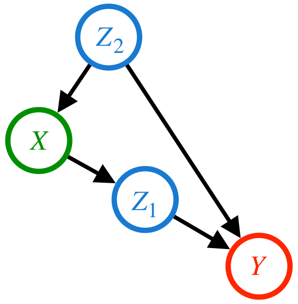


$$P(y|\, do(x)) = \sum_{z_1} P(y \mid x, z_1) P(z_1)$$

Backdoor Criterion

Related work

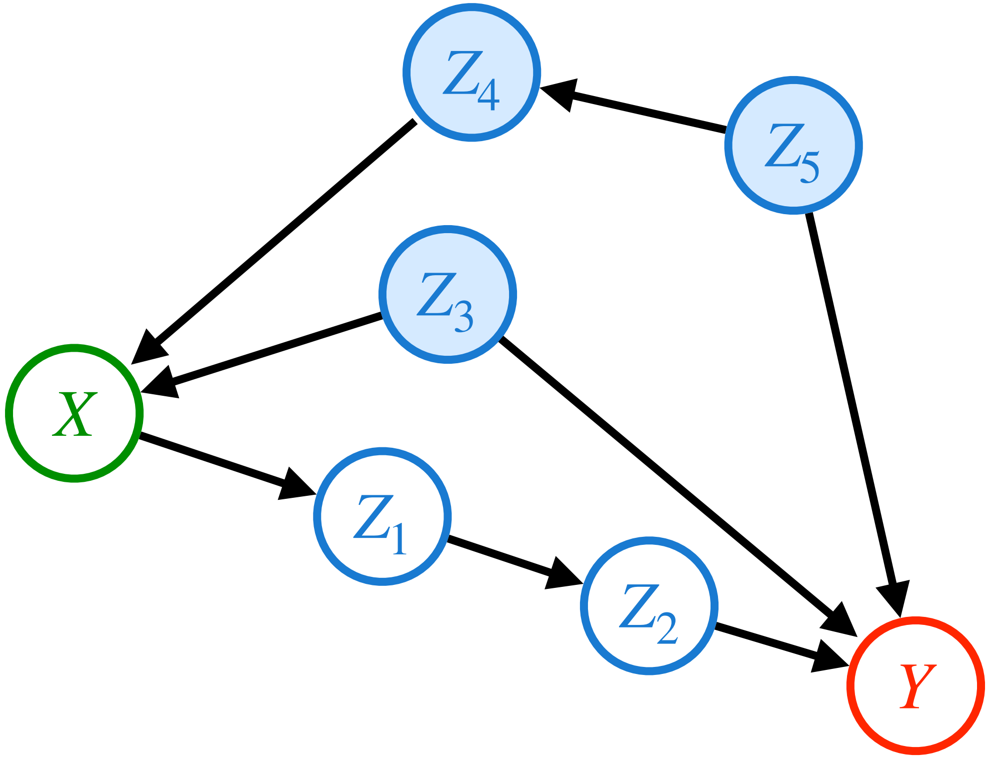
Backdoor adjustment criterion

given a graph  and a covariate set $\mathbf{Z}$

Backdoor adjustment criterion

whether $P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})$

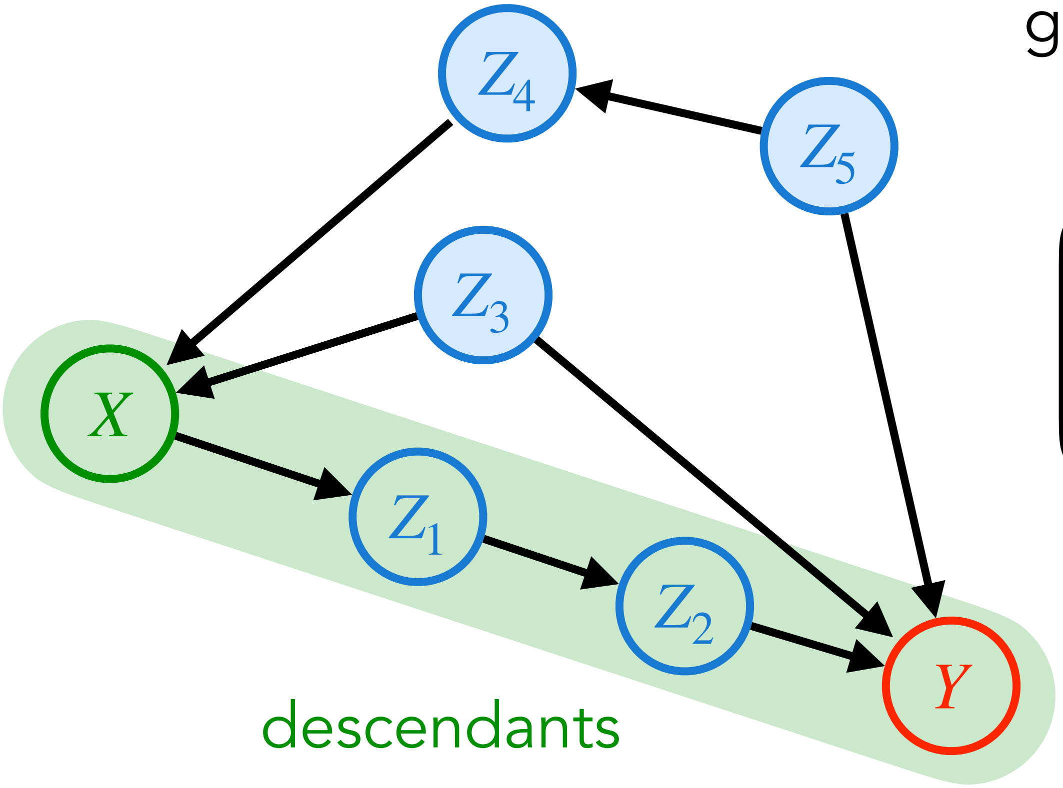
Backdoor adjustment criterion



given a graph and a covariate set $\mathbf{Z} = \{Z_3, Z_4, Z_5\}$

1. $Z \in \mathbf{Z}$ is *not* a descendant of $\mathbf{X}$.
2. $\mathbf{Z}$ blocks every backdoor path.

Backdoor adjustment criterion



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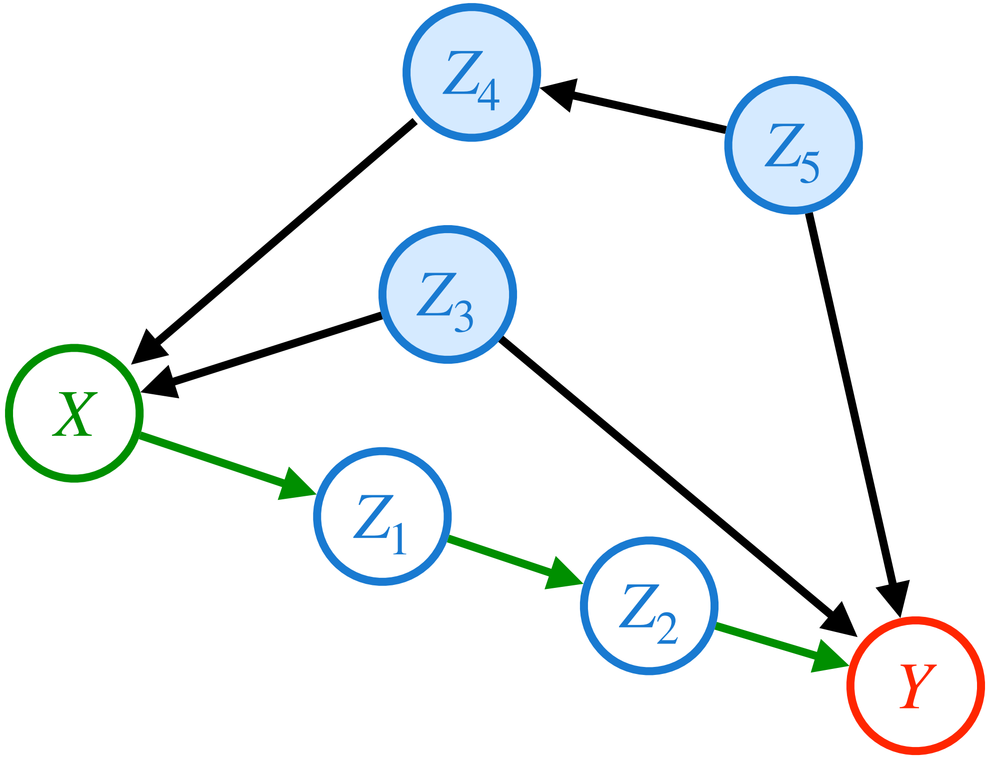

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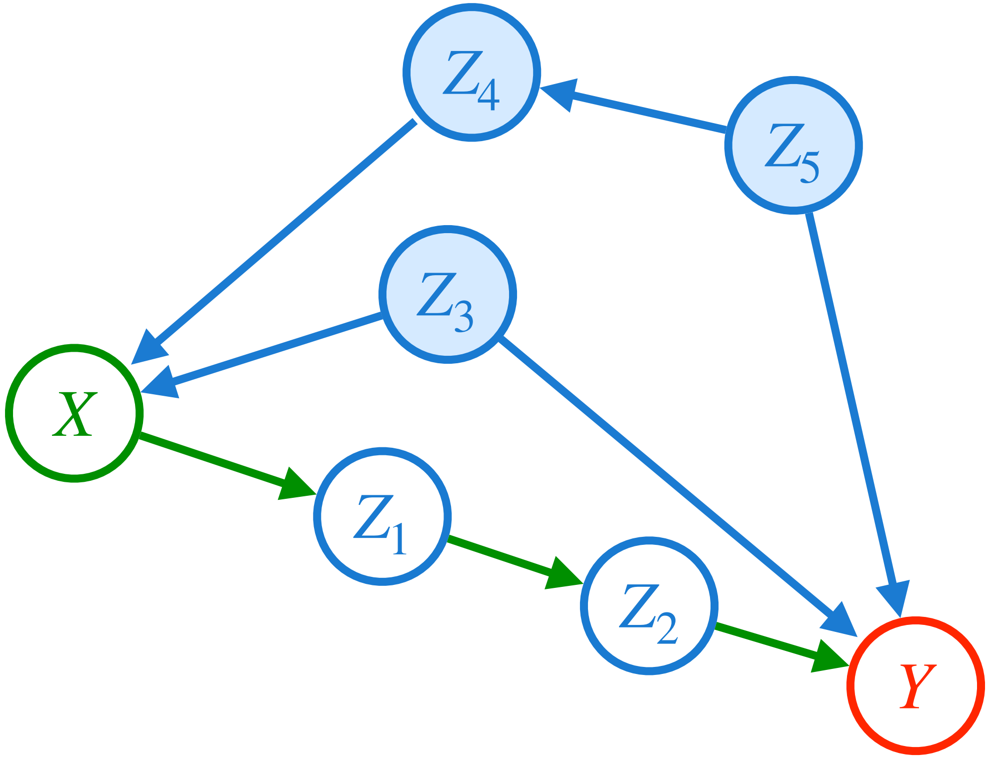
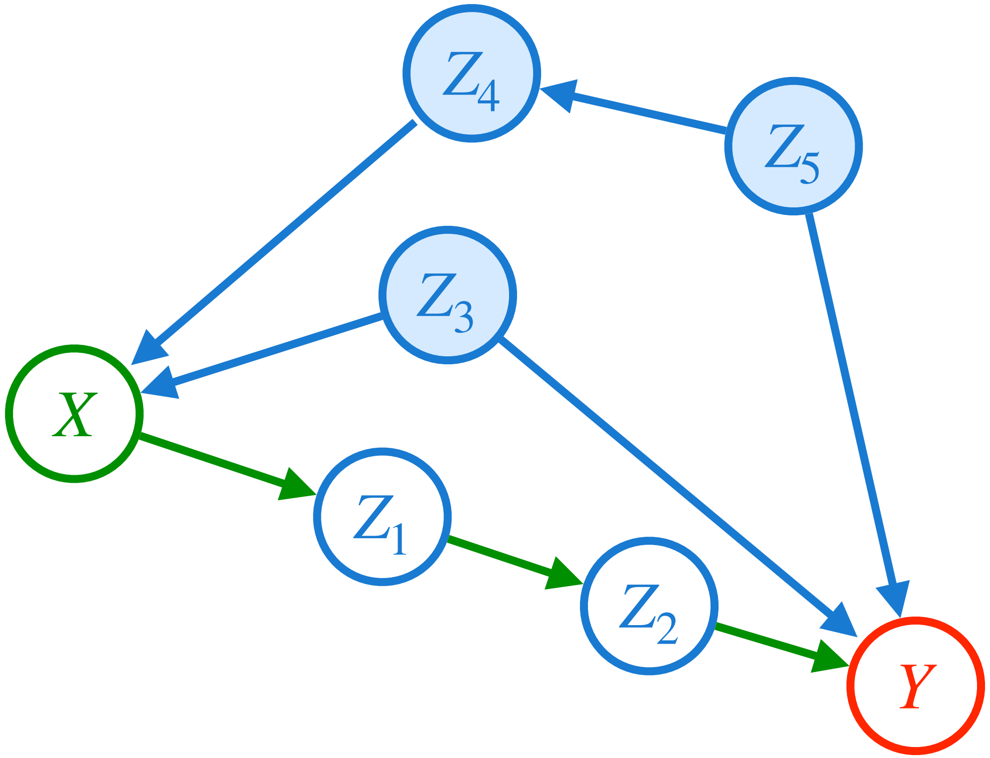


given a graph and a covariate set $\mathbf{Z} = \{Z_3, Z_4, Z_5\}$

✔ 1. $Z \in \mathbf{Z}$ is *not* a descendant of $\mathbf{X}$.

2. $\mathbf{Z}$ blocks every backdoor path.

Backdoor adjustment criterion



given a graph and a covariate set $\mathbf{Z} = \{Z_3, Z_4, Z_5\}$

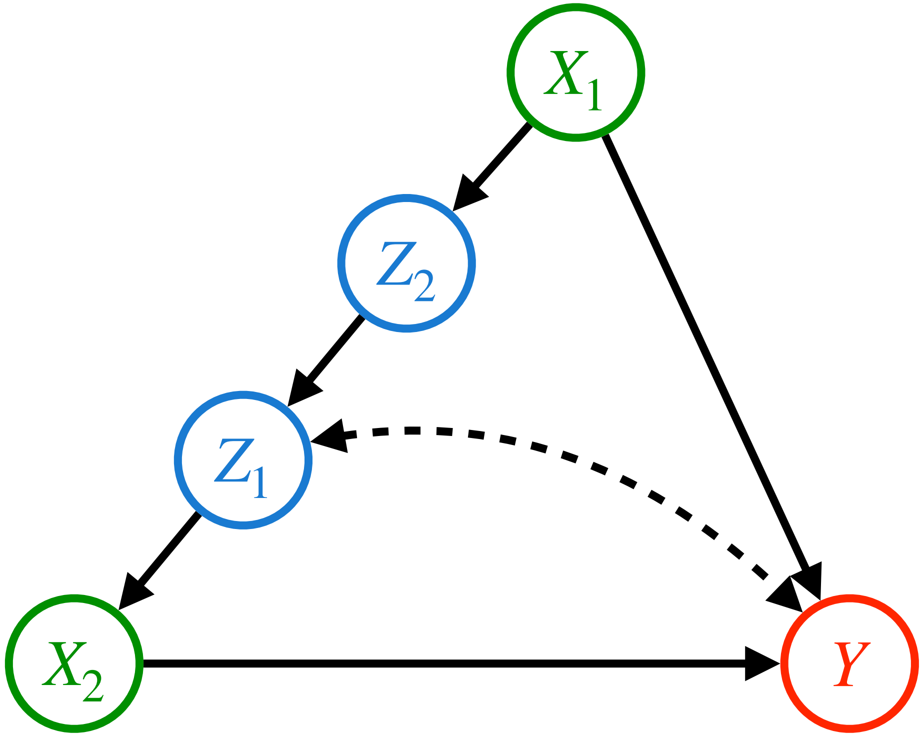
✓ 1. $Z \in \mathbf{Z}$ is *not* a descendant of $\mathbf{X}$.

✓ 2. $\mathbf{Z}$ blocks every backdoor path.

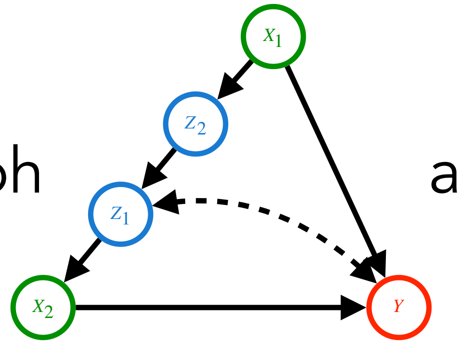
$$P(y \mid do(x)) = \sum_{\mathbf{z}} P(y \mid x, z_3, z_4, z_5) P(z_3, z_4, z_5)$$

Backdoor adjustment criterion - sufficient but not necessary

Backdoor criterion is sufficient, but *not* necessary for Covariate Adjustment.



Backdoor adjustment criterion - sufficient but not necessary

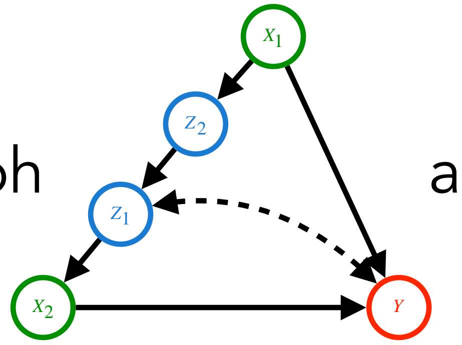
given a graph  and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

Backdoor adjustment criterion

Return *False*.

Backdoor adjustment criterion - sufficient but not necessary

given a graph



and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

Backdoor adjustment criterion

Return *False*.

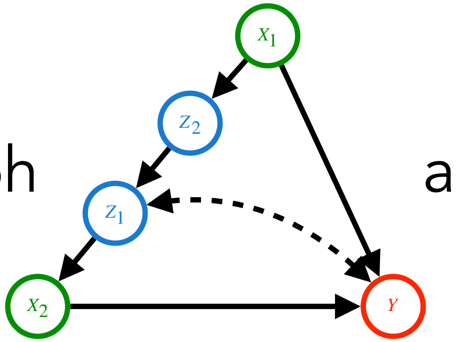
However $P(y|\, do(x)) = \sum_{\mathbf{z}} P(y \mid x, z_1, z_2) P(z_1, z_2)$ holds!

Adjustment Criterion

Related work

Adjustment criterion

given a graph

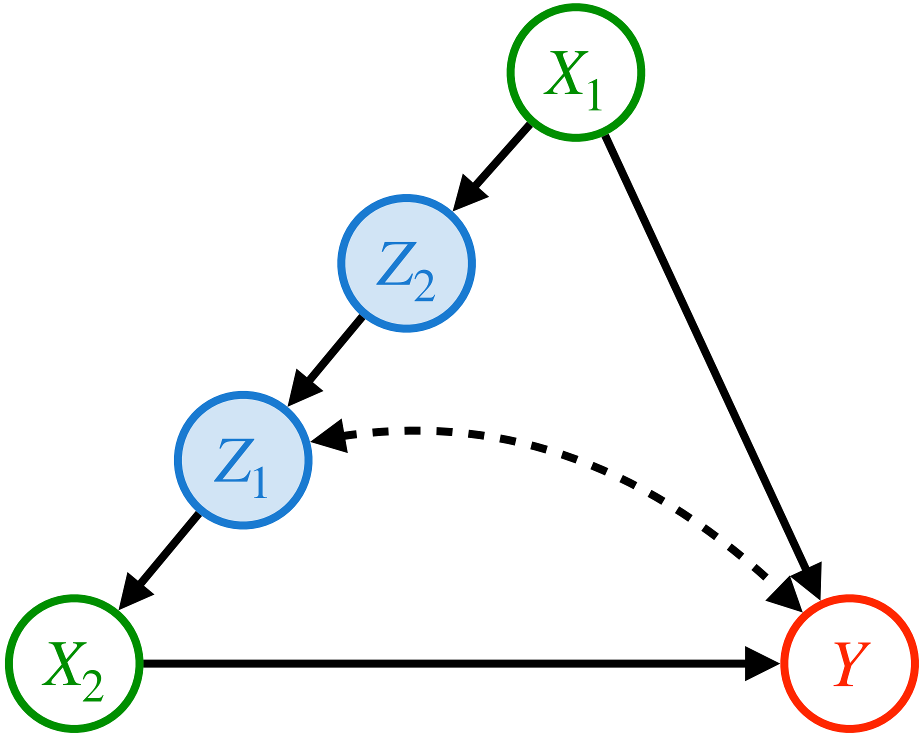


and a covariate set $\mathbf{Z}$

Adjustment criterion

whether $P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z})$

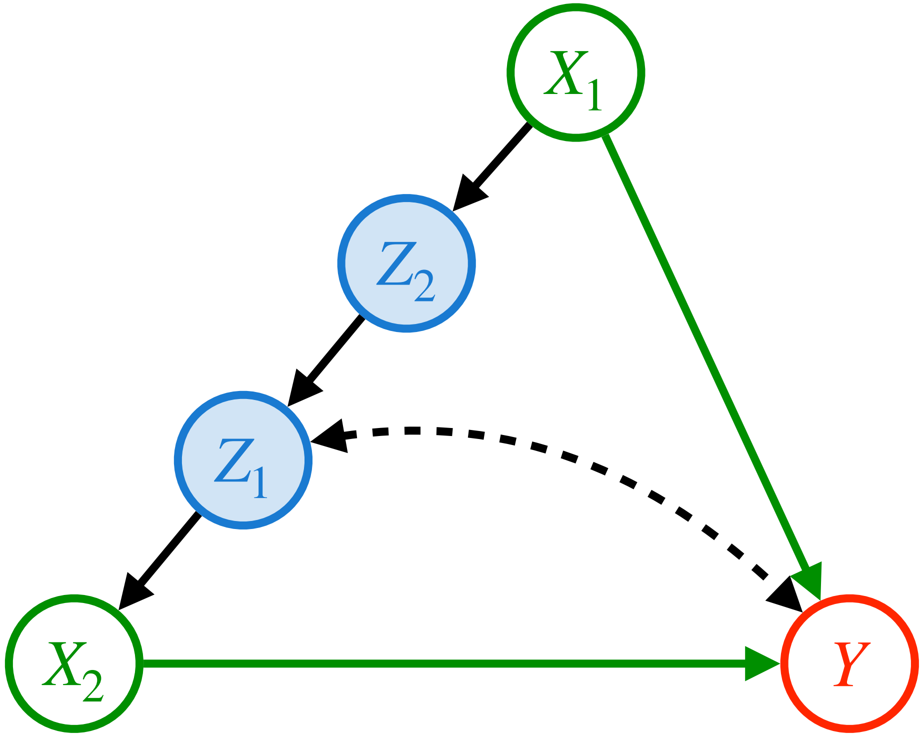
Adjustment criterion



given a graph and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

1. $\mathbf{Z}$ does *not* disturb all proper causal paths.
2. $\mathbf{Z}$ blocks all paths in the subgraph.

Adjustment criterion

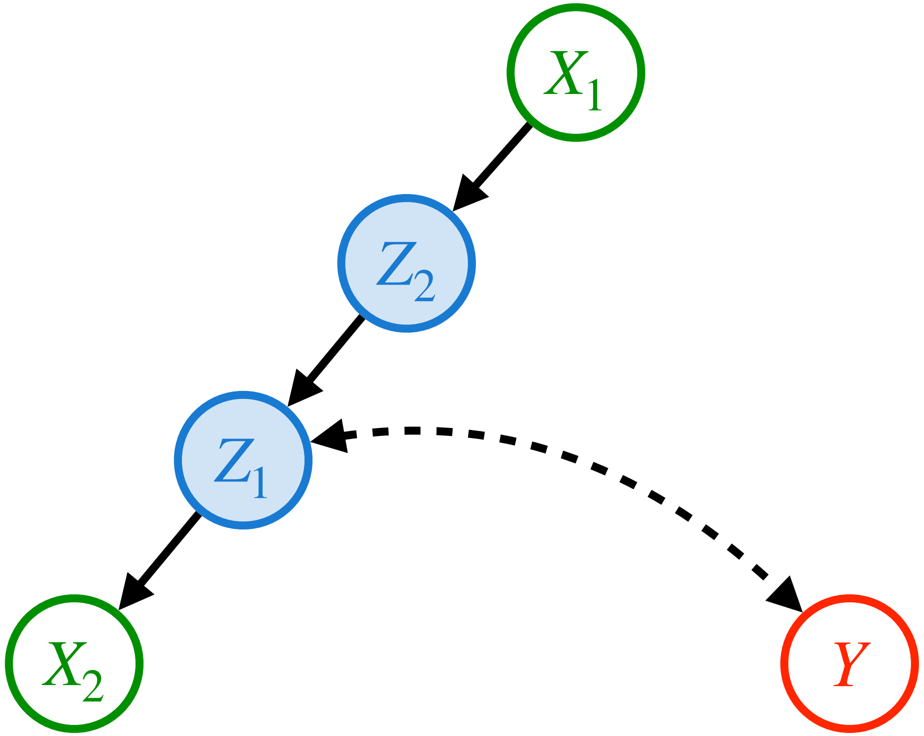


given a graph and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

1. $\mathbf{Z}$ does *not disturb* all proper causal paths.
2. $\mathbf{Z}$ blocks all paths in the subgraph.

Proper causal path $X_1 \to Y$ and $X_2 \to Y$.

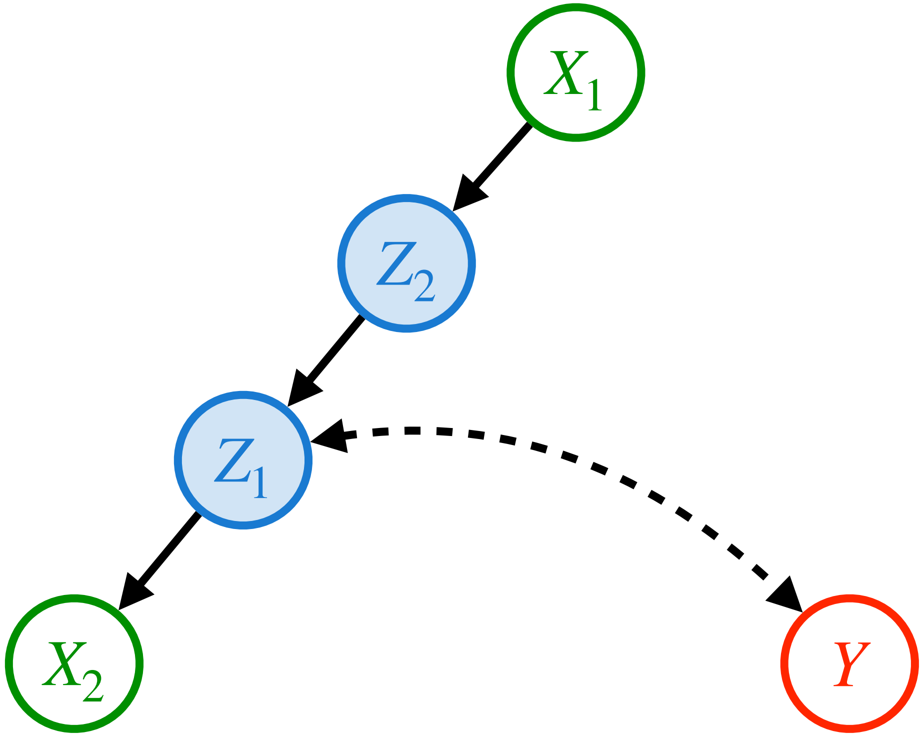
Adjustment criterion



given a graph and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

1. $\mathbf{Z}$ does *not disturb* all proper causal paths. ✔
2. $\mathbf{Z}$ blocks all paths in the subgraph.

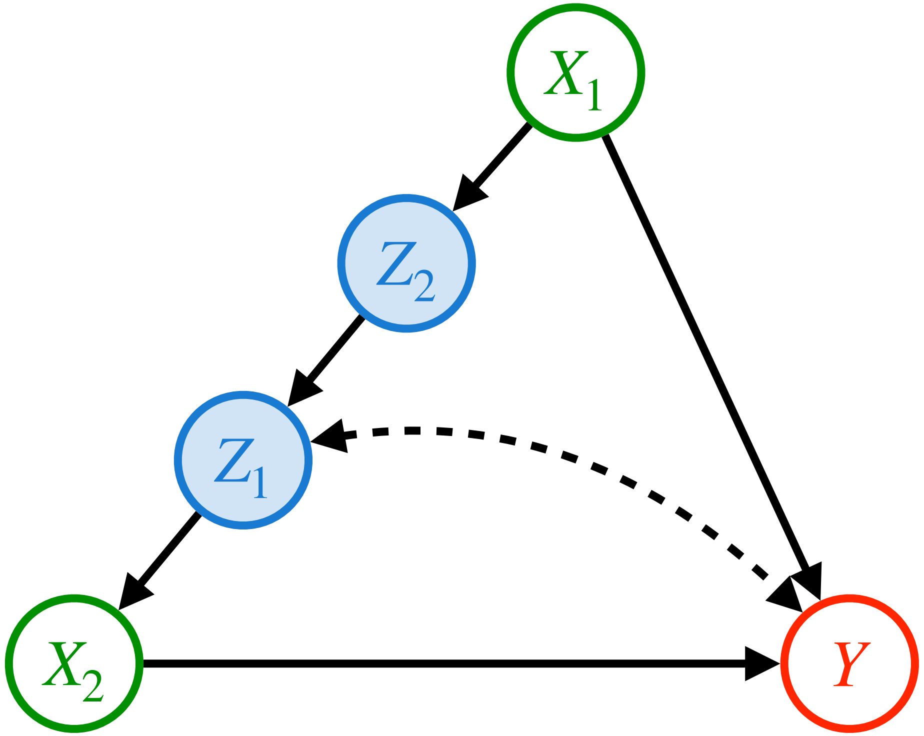
Adjustment criterion



given a graph and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

1. ✓ $\mathbf{Z}$ does *not* disturb all proper causal paths.
2. ✓ $\mathbf{Z}$ blocks all paths in the subgraph.

Adjustment criterion

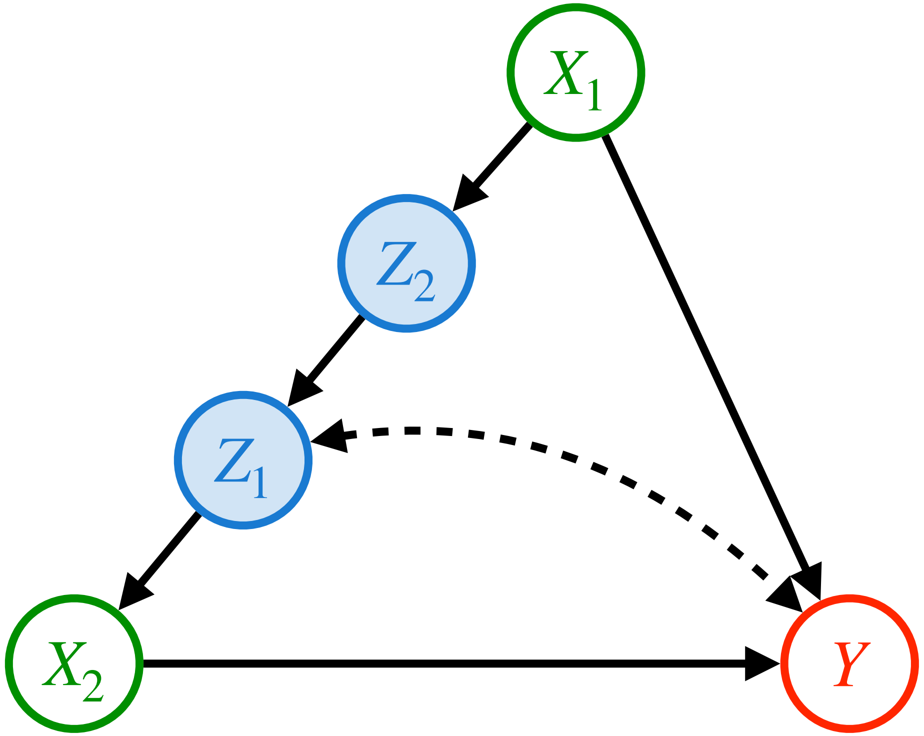


given a graph and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

1. ✓ $\mathbf{Z}$ does *not disturb* all proper causal paths.
2. ✓ $\mathbf{Z}$ blocks all paths in the subgraph.

$$P(y \mid do(x)) = \sum_{\mathbf{z}} P(y \mid x, z_1, z_2) P(z_1, z_2)$$

Adjustment criterion



given a graph and a covariate set $\mathbf{Z} = \{Z_1, Z_2\}$

$\Uparrow$

1. $\mathbf{Z}$ does *not disturb* all proper causal paths.
2. $\mathbf{Z}$ blocks all paths in the subgraph.

$\Downarrow$

Completeness!

$$P(y | do(x)) = \sum_{\mathbf{z}} P(y \mid x, z_1, z_2) P(z_1, z_2)$$

Sequential Adjustment Criterion

Main Theory

Sequential covariate adjustment

Covariate Adjustment (CA)

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{z}) P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})$$

Sequential covariate adjustment

Covariate Adjustment (CA)

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{z}) P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})$$

Sequential Covariate Adjustment (SCA)

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} \prod_{j=0}^{m} P(\mathbf{z}_{j+1}, \mathbf{y}_j \mid \mathbf{x}_{j-1}, \mathbf{y}_{j-1}, \mathbf{z}_{j-1}, \mathbf{x}_j, \mathbf{z}_j) P(\mathbf{z}_j)$$

Sequential covariate adjustment

Covariate Adjustment (CA)

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{z}) P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})$$

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$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} \prod_{j=0}^{m} P(\mathbf{z}_{j+1}, \mathbf{y}_j \mid \mathbf{x}_{j-1}, \mathbf{y}_{j-1}, \mathbf{z}_{j-1}, \mathbf{x}_j, \mathbf{z}_j) P(\mathbf{z}_j)$$

e.g. $P(\mathbf{r} \mid do(\mathbf{a})) = \sum P(\mathbf{s}_1) P(\mathbf{r}_1, \mathbf{s}_2 \mid \mathbf{a}_1, \mathbf{s}_1) P(\mathbf{r}_2, \mathbf{s}_3 \mid \mathbf{a}_1, \mathbf{r}_1, \mathbf{s}_1, \mathbf{a}_2, \mathbf{s}_2) \cdots$

Sequential covariate adjustment

Covariate Adjustment (CA)

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} P(\mathbf{z}) P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})$$

CA is a special case of SCA where $m = 1$

SCA is generalized version of CA!

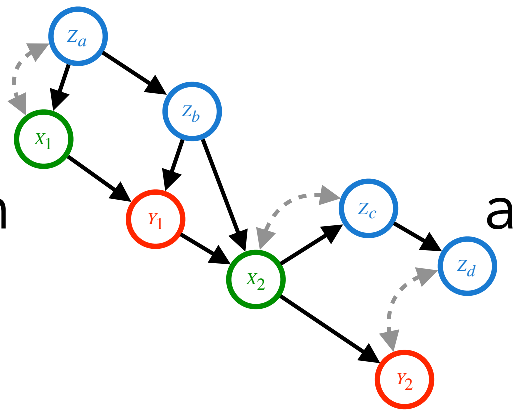
Sequential Covariate Adjustment (SCA)

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} \prod_{j=0}^{m} P(\mathbf{z}_{j+1}, \mathbf{y}_j \mid \mathbf{x}_{j-1}, \mathbf{y}_{j-1}, \mathbf{z}_{j-1}, \mathbf{x}_j, \mathbf{z}_j) P(\mathbf{z}_j)$$

e.g. $P(\mathbf{r} \mid do(\mathbf{a})) = \sum P(\mathbf{s}_1) P(\mathbf{r}_1, \mathbf{s}_2 \mid \mathbf{a}_1, \mathbf{s}_1) P(\mathbf{r}_2, \mathbf{s}_3 \mid \mathbf{a}_1, \mathbf{r}_1, \mathbf{s}_1, \mathbf{a}_2, \mathbf{s}_2) \cdots$

Sequential adjustment criterion

given a graph



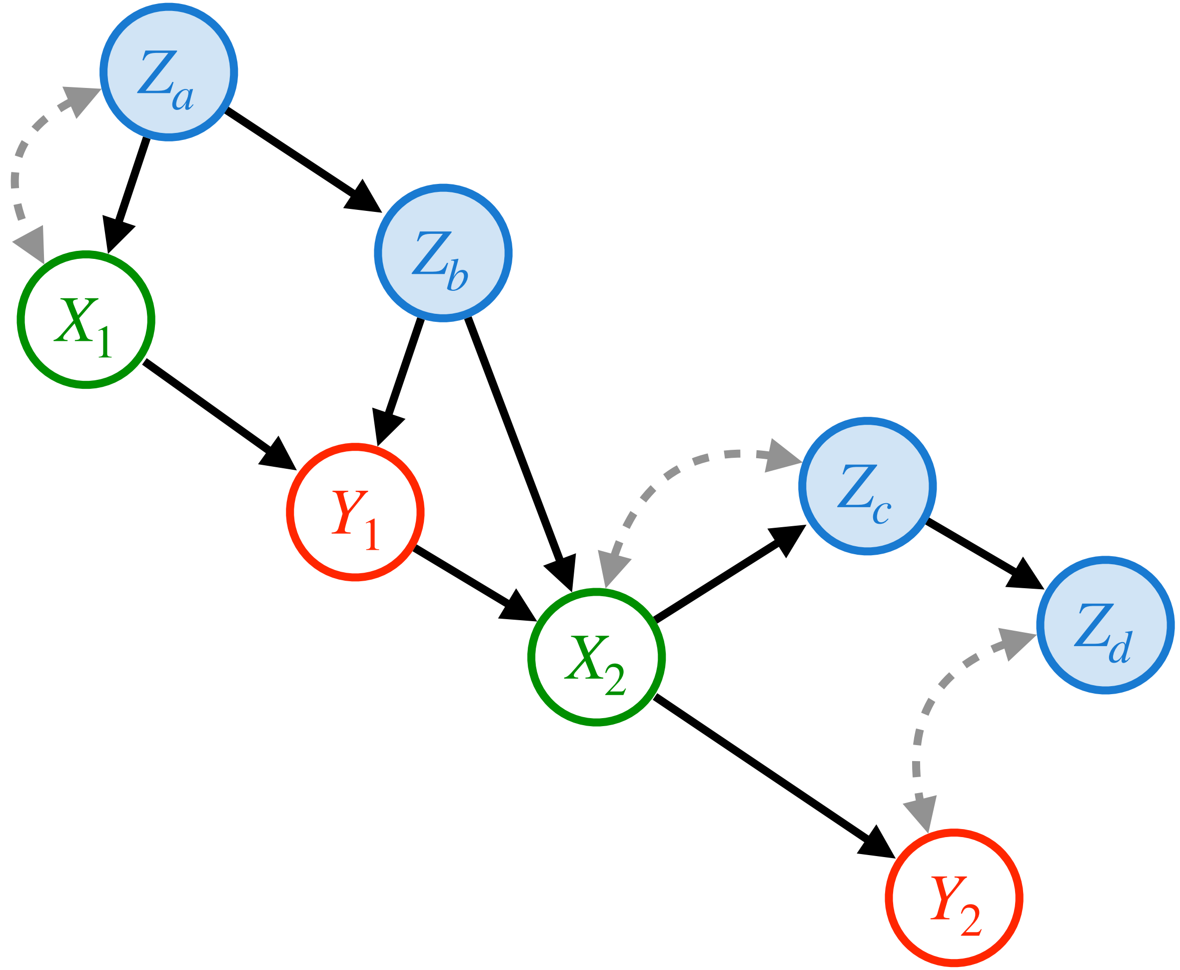
and a covariate set $\mathbf{Z} = (\{Z_a, Z_b\}, \{Z_c, Z_d\})$

$\Updownarrow$

Sequential adjustment criterion
(SAC)

whether $P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} \prod_{j=0}^{m} P(\mathbf{z}_{j+1}, \mathbf{y}_j \mid \mathbf{x}_{j-1}, \mathbf{y}_{j-1}, \mathbf{z}_{j-1}, \mathbf{x}_j, \mathbf{z}_j) P(\mathbf{z}_j)$

Sequential adjustment criterion



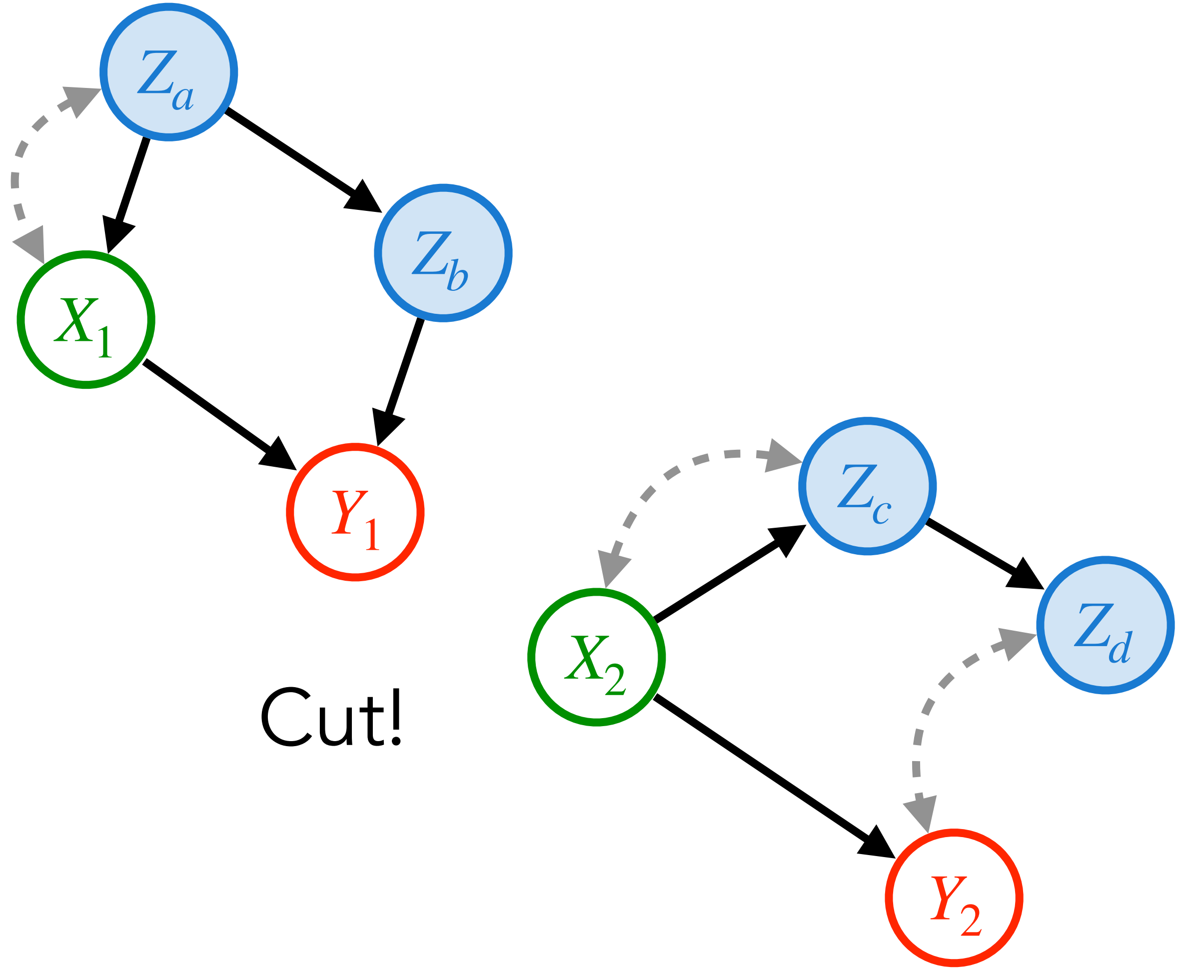
given a graph and a covariate set

$\mathbf{Z} = (\{Z_a, Z_b\}, \{Z_c, Z_d\})$

1. $\mathbf{Z}_1 = \{Z_a, Z_b\}$ does *not disturb* all local proper causal paths.

2. $\mathbf{Z}_1 = \{Z_a, Z_b\}$ blocks all paths in the subgraph.

Sequential adjustment criterion



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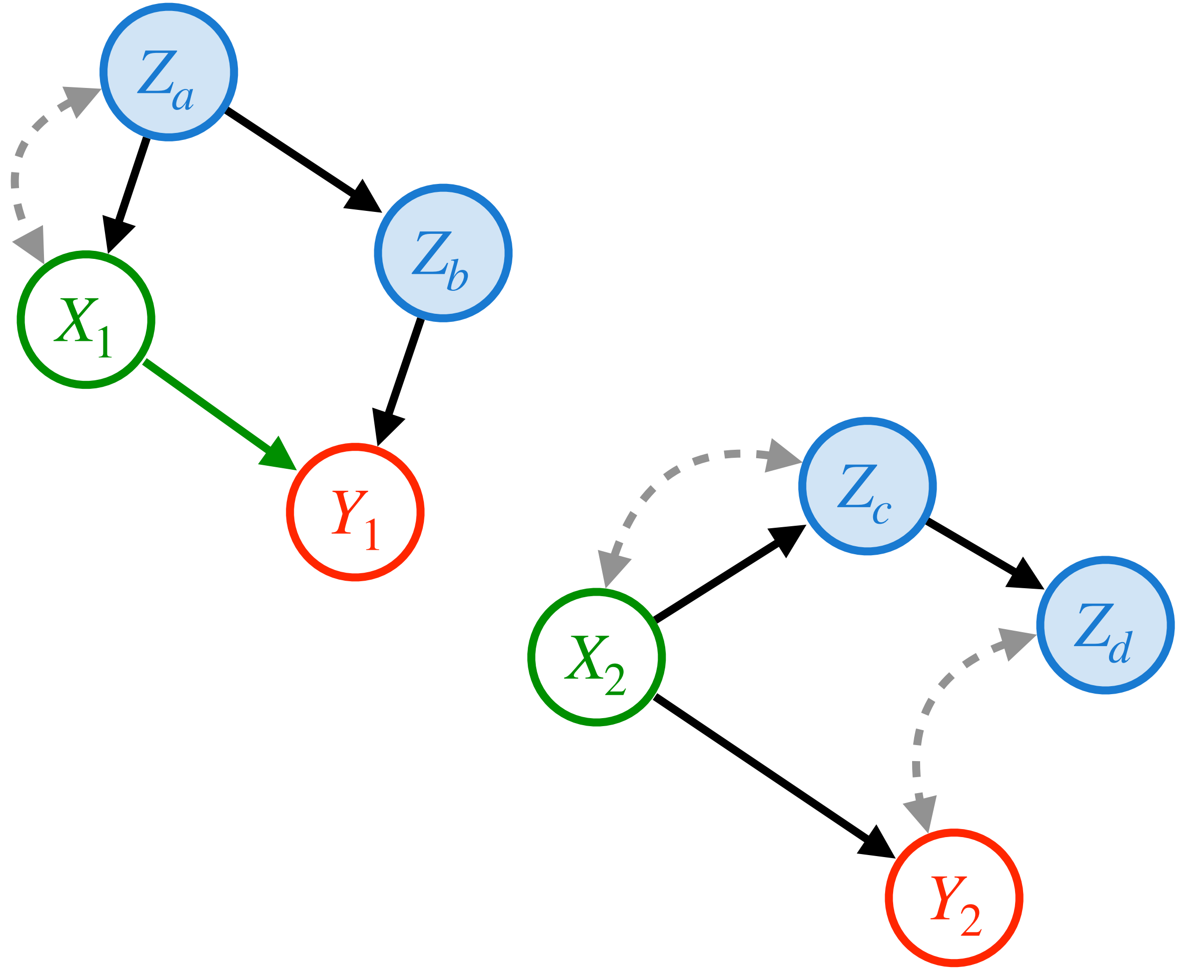

given a graph and a covariate set

$\mathbf{Z} = (\{Z_a, Z_b\}, \{Z_c, Z_d\})$

1. $\mathbf{Z}_1 = \{Z_a, Z_b\}$ does *not disturb* all local proper causal paths.

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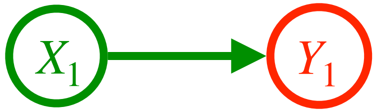
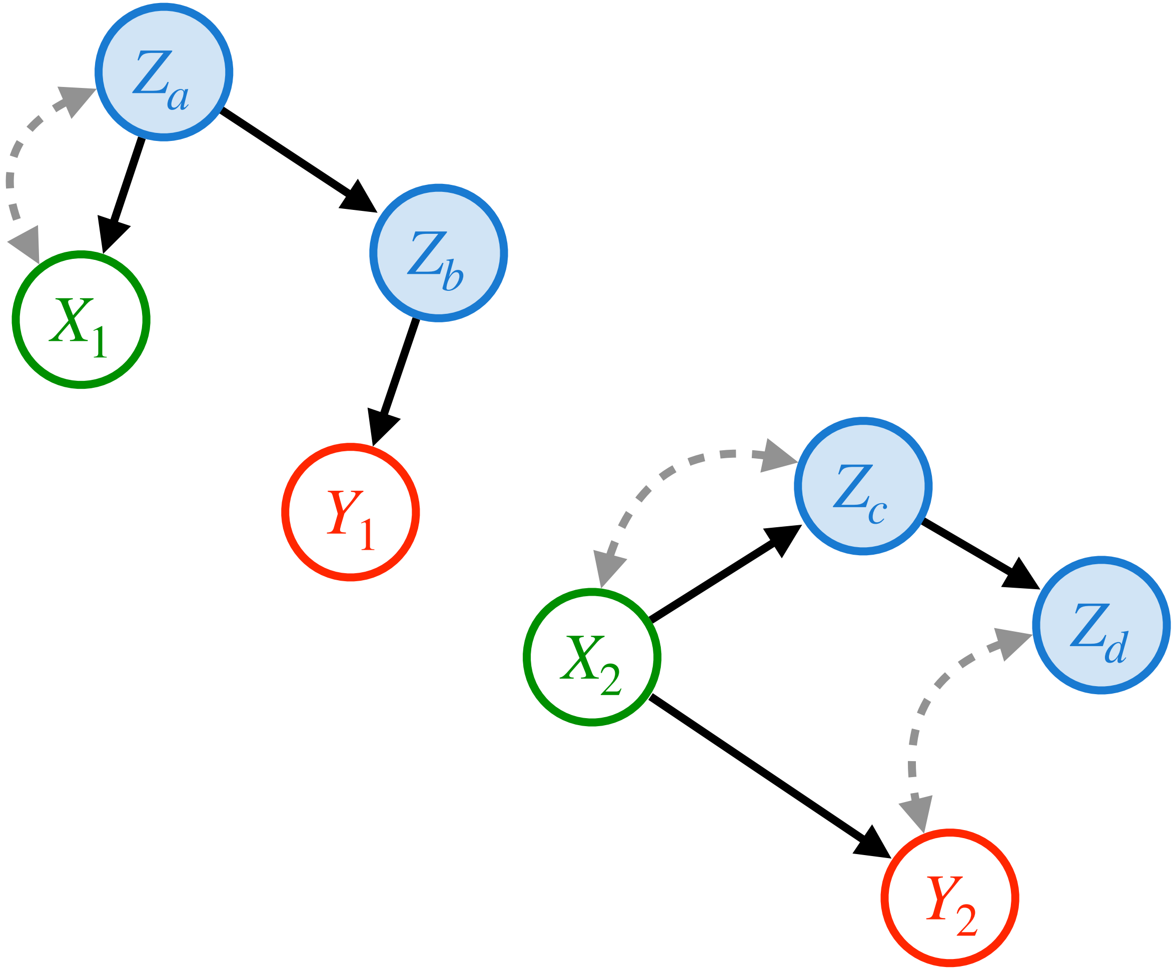
Proper causal path  .

Sequential adjustment criterion



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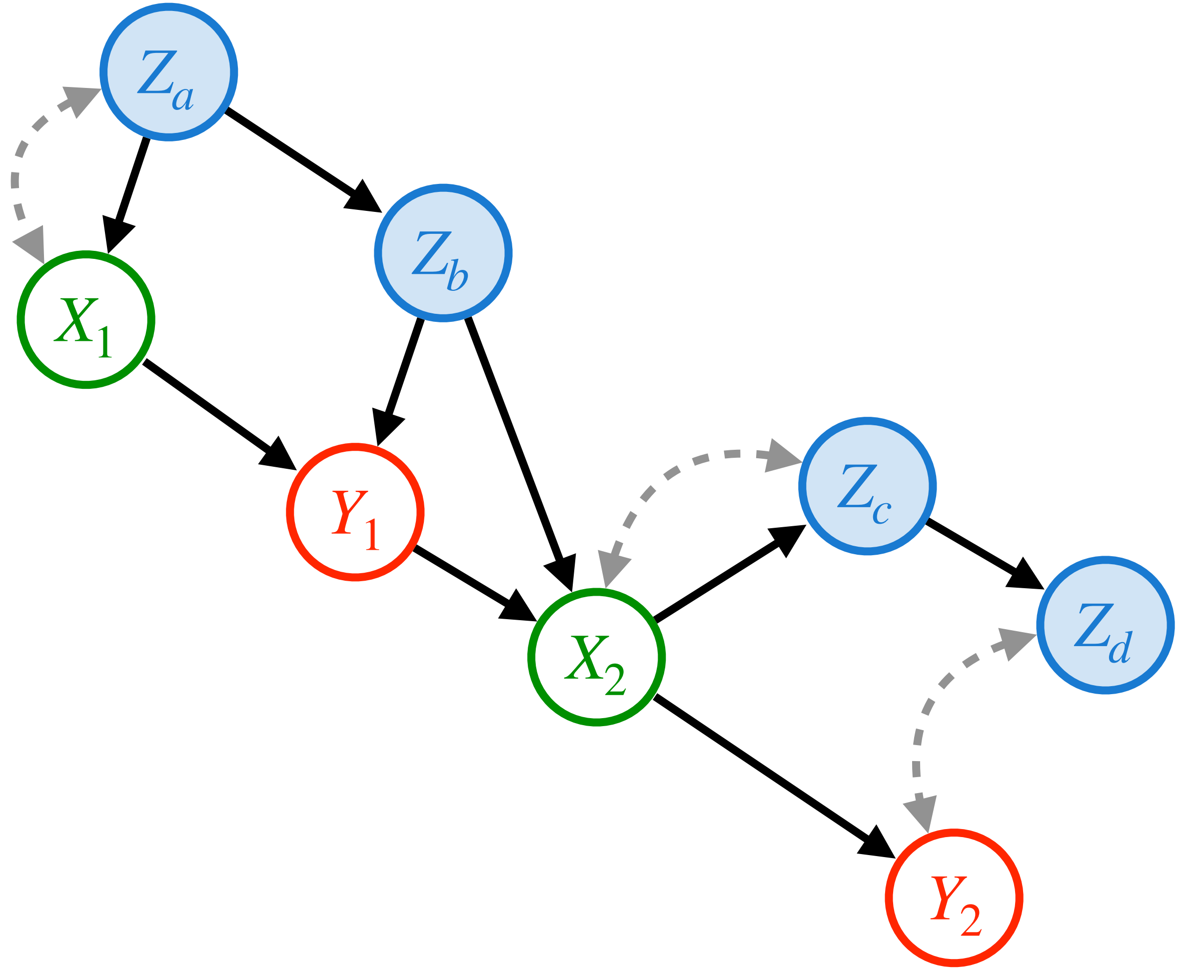
Sequential adjustment criterion



given a graph and a covariate set

$$\mathbf{Z} = (\{Z_a, Z_b\}, \{Z_c, Z_d\})$$

1. $\mathbf{Z}_2 = \{Z_c, Z_d\}$ does *not disturb* all local proper causal paths.
2. $\mathbf{Z}_2 = \{Z_c, Z_d\}$ blocks all paths in the subgraph.

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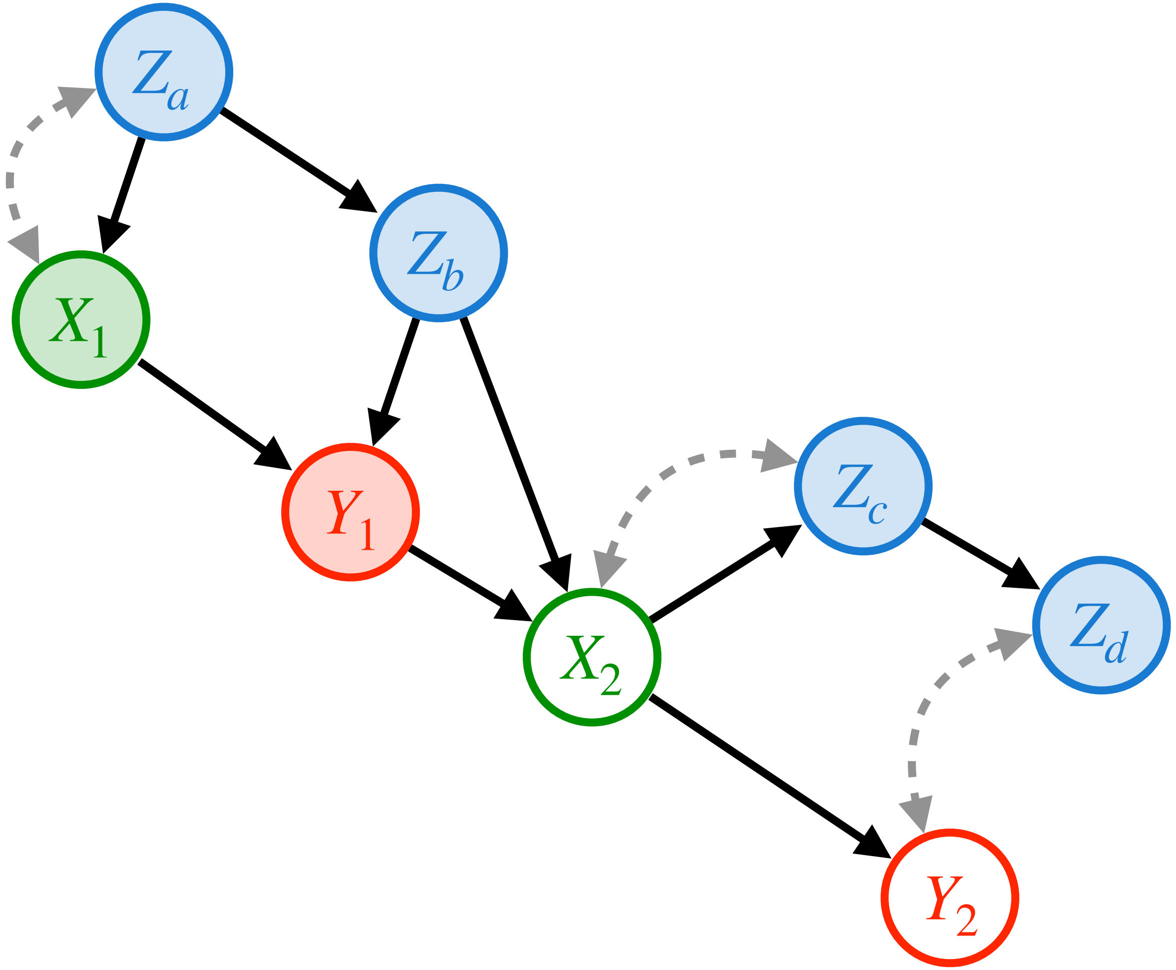

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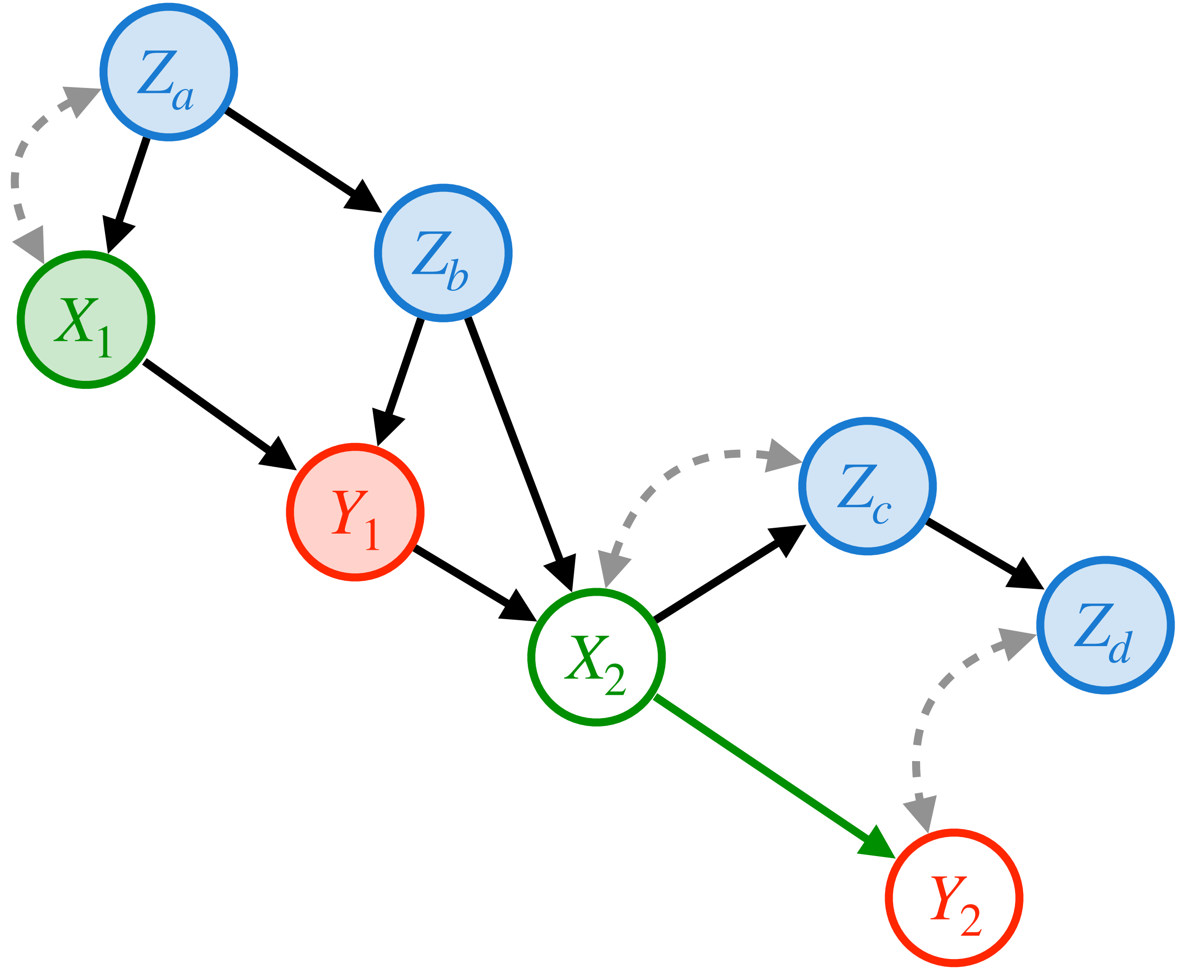

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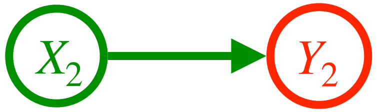
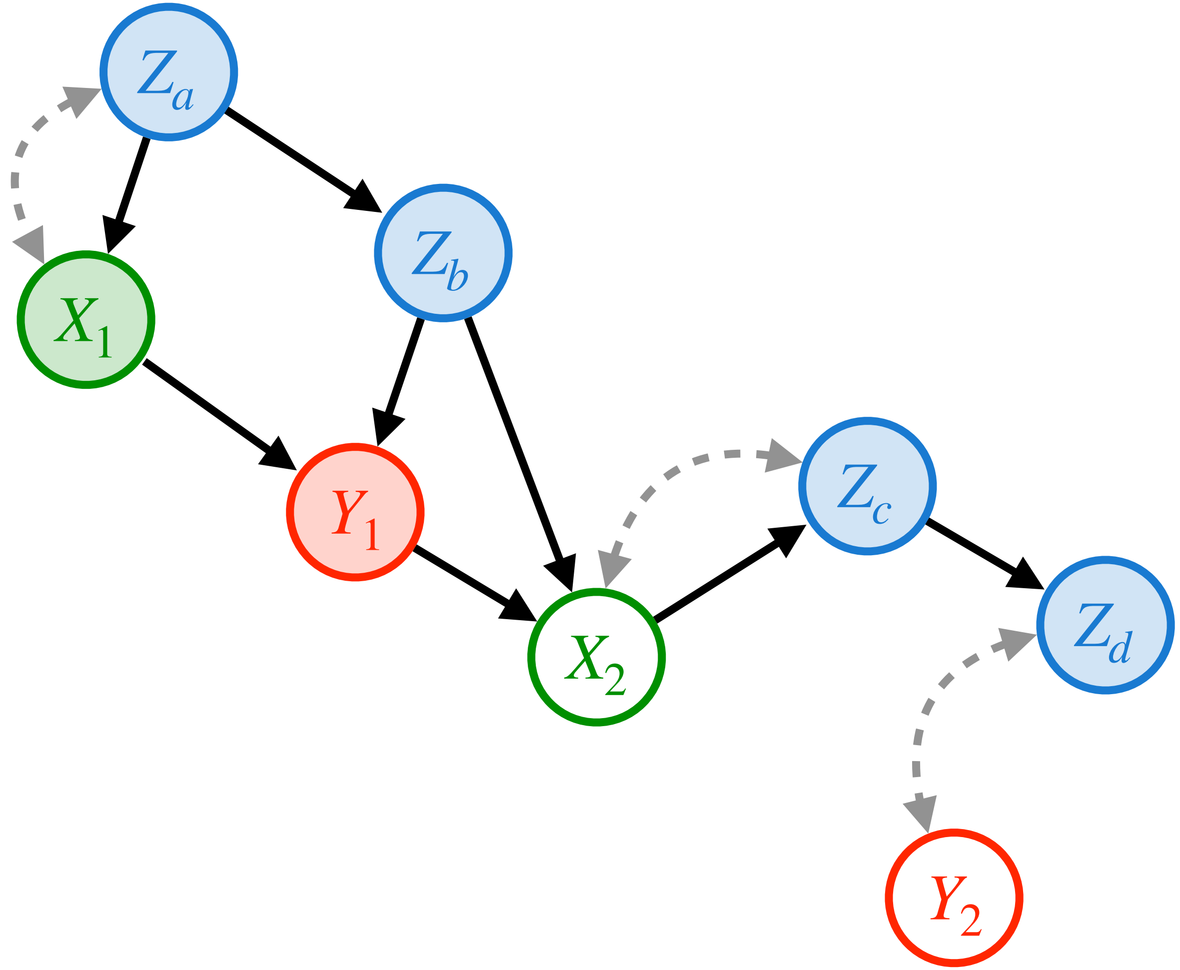
Proper causal path  .

Sequential adjustment criterion



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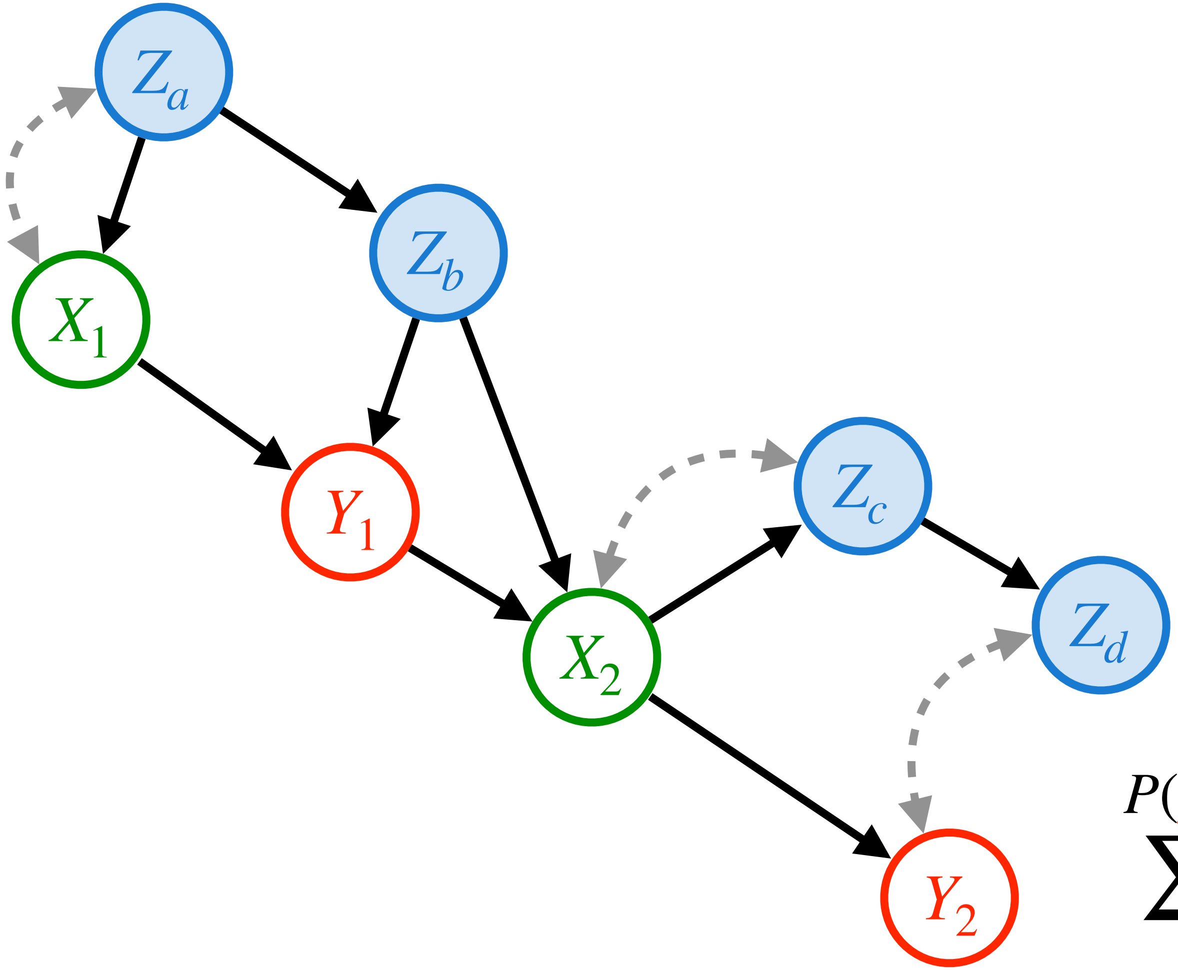


given a graph and a covariate set

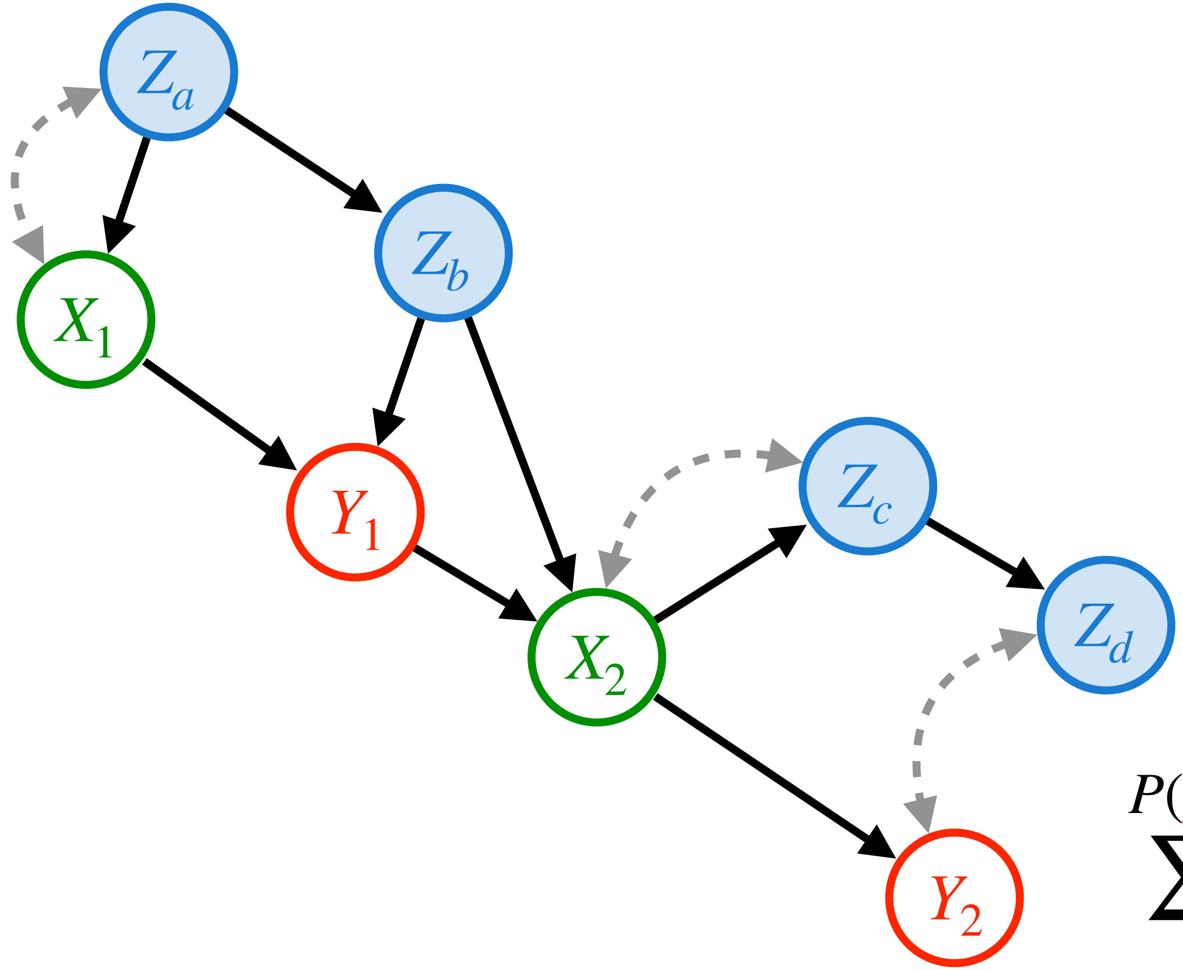
$$\mathbf{Z} = (\{Z_a, Z_b\}, \{Z_c, Z_d\})$$

1. $\mathbf{Z}_j$ does *not disturb* all local proper causal paths.

2. $\mathbf{Z}_j$ blocks all paths in the subgraph.

$$P(y_1, y_2 \mid do(x_1, x_2)) =$$
$$\sum P(z_a, z_b) P(y_1, z_c, z_d \mid x_1, z_a, z_b) P(y_2 \mid x_1, y_1, z_a, z_b, z_c, z_d)$$

Sequential adjustment criterion



given a graph and a covariate set

$$\mathbf{Z} = (\{Z_a, Z_b\}, \{Z_c, Z_d\})$$

$\Updownarrow$

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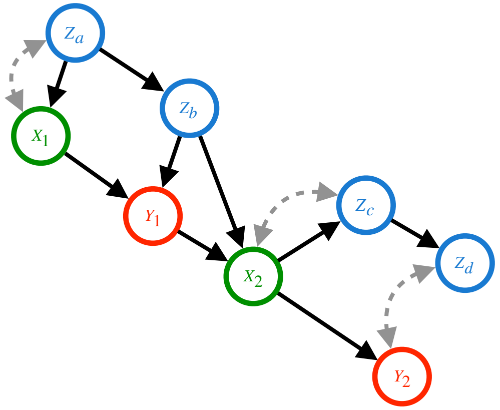
Completeness!

$$P(y_1, y_2 \mid do(x_1, x_2)) =$$
$$\sum P(z_a, z_b) P(y_1, z_c, z_d \mid x_1, z_a, z_b) P(y_2 \mid x_1, y_1, z_a, z_b, z_c, z_d)$$

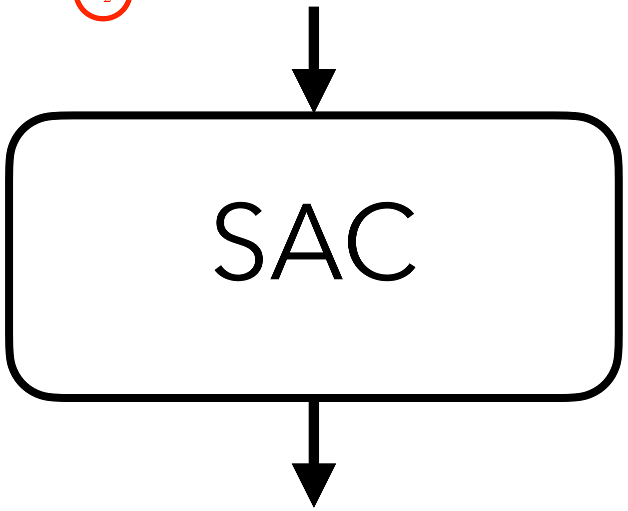
minSCA

Main Theory

Constuctive SCA algorithm

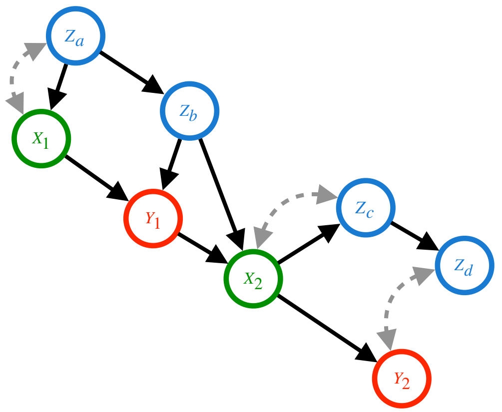


and $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$



whether $P(\mathbf{y} \mid do(\mathbf{x})) =$ SCA?

Constuctive SCA algorithm



and $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$

↓

SAC

↓

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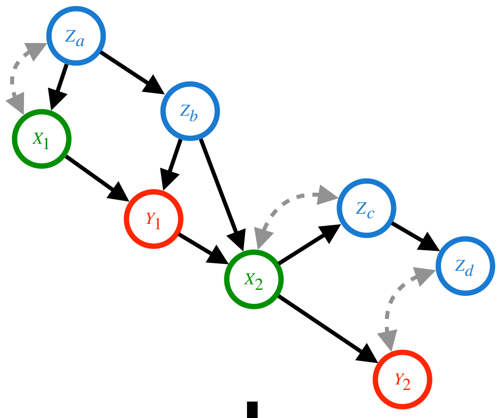


↓

minSCA

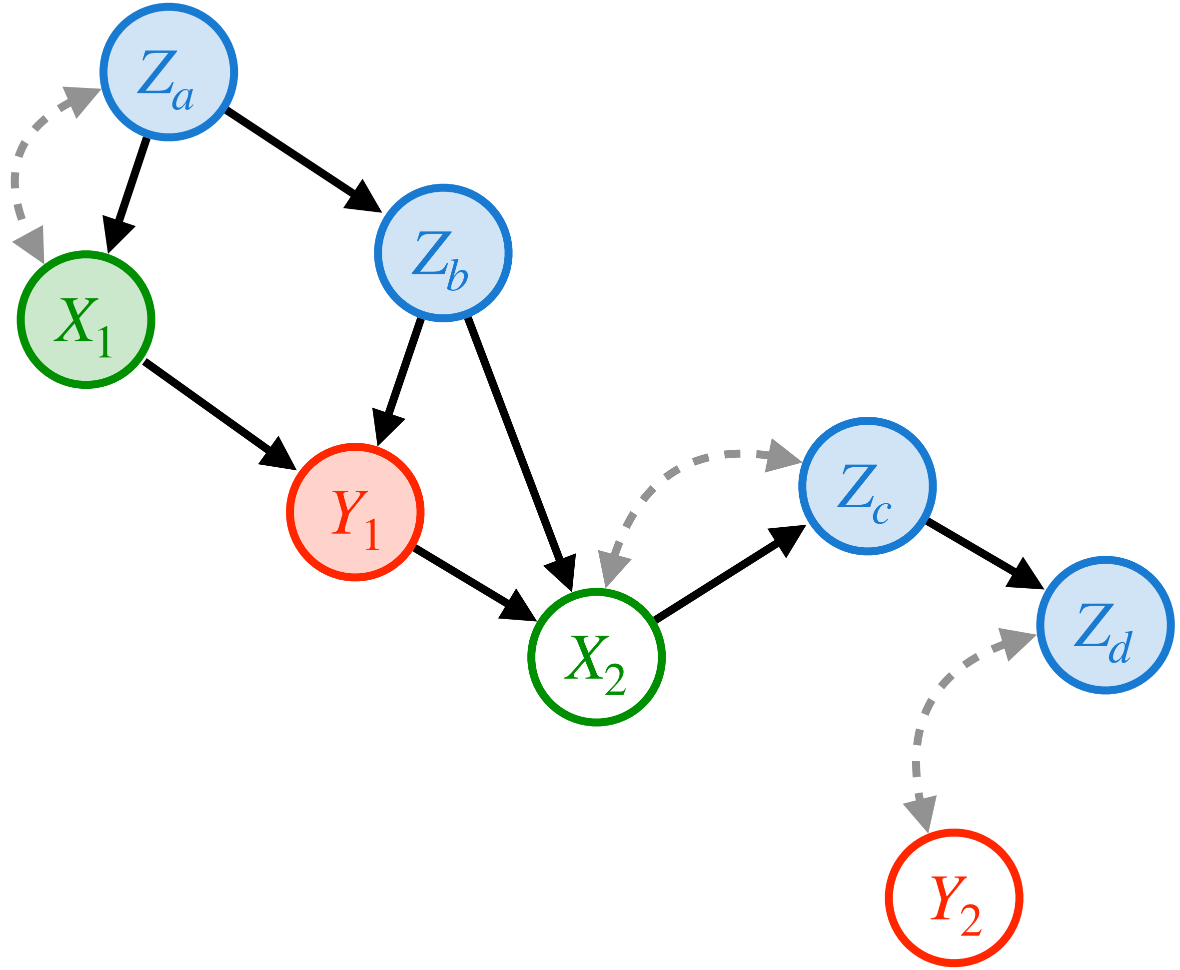
↓

$\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$

possibly satisfying

$P(\mathbf{y} \mid do(\mathbf{x}))$ = SCA

Sequential adjustment criterion



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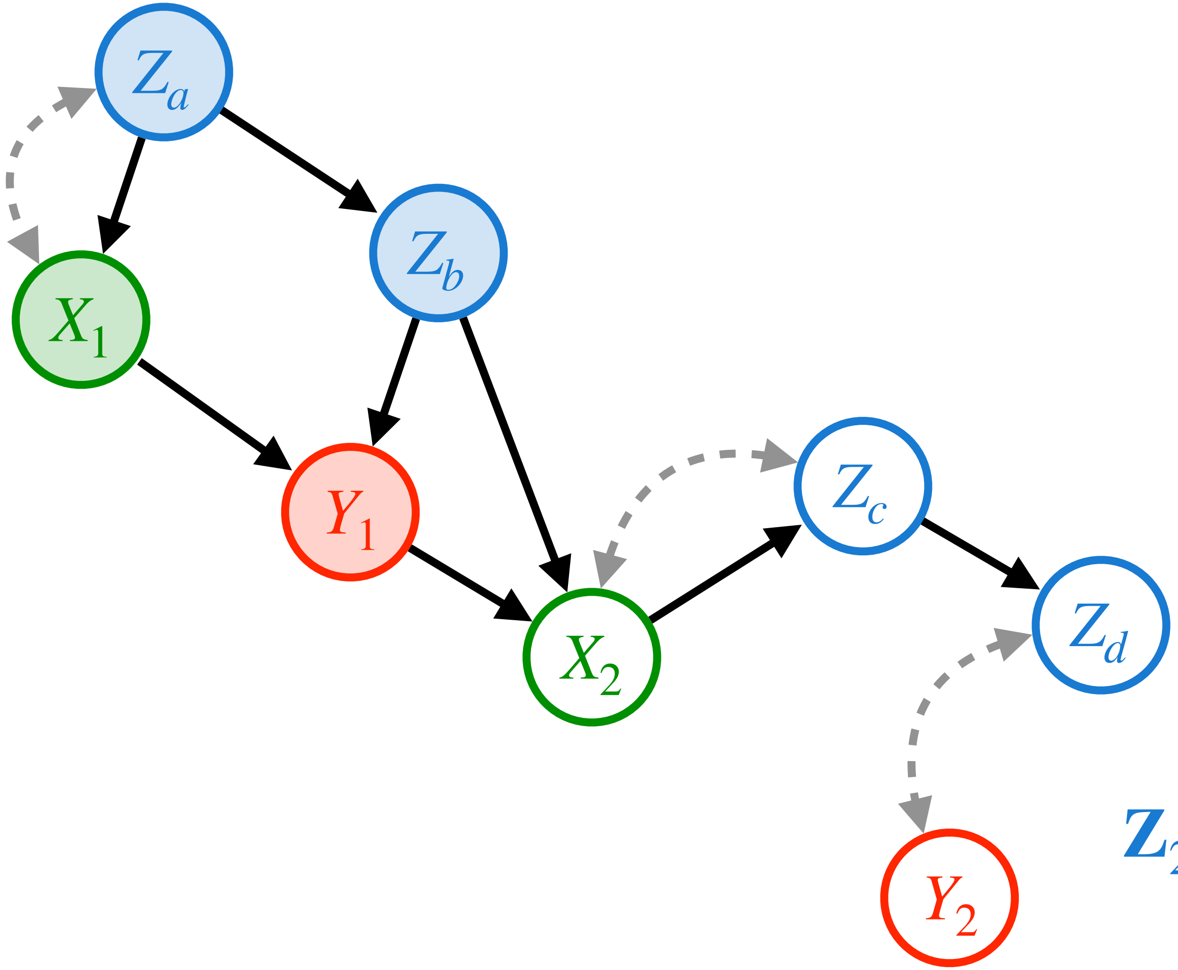

given a graph and a covartiate set

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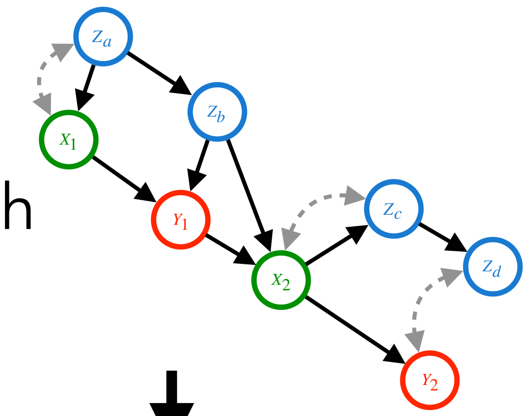
2. $\mathbf{Z}_2 = \{Z_c, Z_d\}$ blocks all paths in the subgraph.

$\mathbf{Z}_2 = \emptyset$ also satisfies the second condition!

Sequential adjustment criterion

given a graph



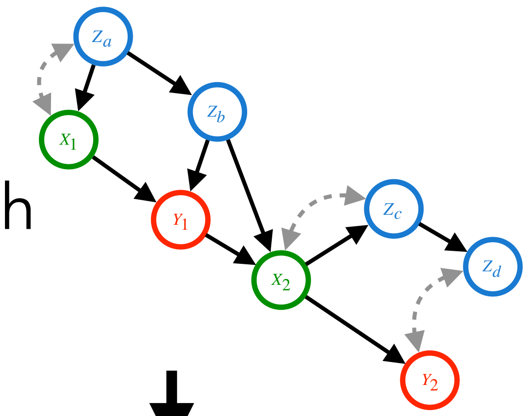
minSCA

minimum size $\mathbf{Z} = (\mathbf{Z}_1, \cdots \mathbf{Z}_m)$ possibly satisfying SCA

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{z}} \prod_{j=0}^{m} P(\mathbf{z}_{j+1}, \mathbf{y}_j \mid \mathbf{x}_{j-1}, \mathbf{y}_{j-1}, \mathbf{z}_{j-1}, \mathbf{x}_j, \mathbf{z}_j) P(\mathbf{z}_j)$$

Sequential adjustment criterion

given a graph



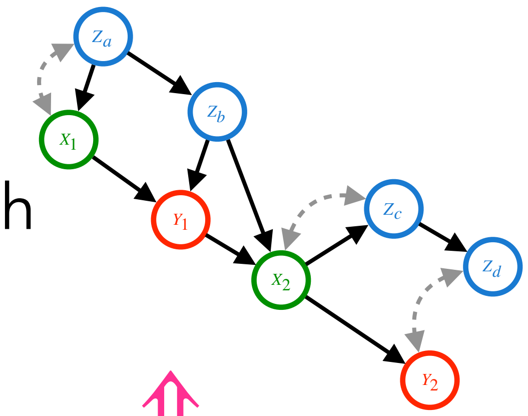
minSCA

$$\mathbf{Z} = (\{Z_a, Z_b\}, \varnothing)$$

$$P(y_1, y_2 \mid do(x_1, x_2)) = \sum P(z_a, z_b) P(y_1 \mid x_1, z_a, z_b) P(y_2 \mid x_1, y_1, z_a, z_b)$$

Sequential adjustment criterion

given a graph



minSCA

$$\mathbf{Z} = (\{Z_a, Z_b\}, \emptyset)$$

$$P(y_1, y_2 \mid do(x_1, x_2)) = \sum P(z_a, z_b) P(y_1 \mid x_1, z_a, z_b) P(y_2 \mid x_1, y_1, z_a, z_b)$$

Conclusion

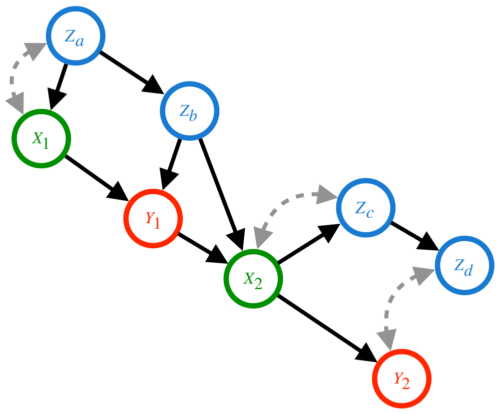
Conclusion

Sequential Adjustment Criterion (SAC), a sound and complete criterion for sequential covariate adjustment.

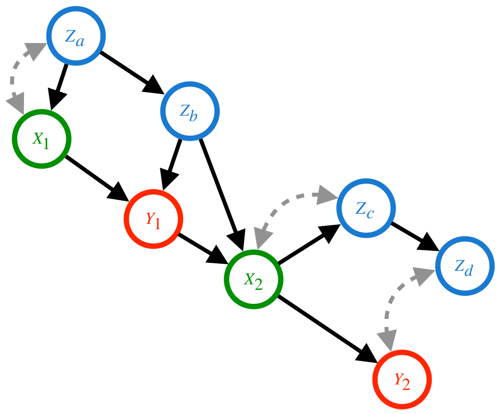
An algorithm **minSCA** for identifying a minimal sequential covariate adjustment set ensuring that no unnecessary vertices are included.

Thank you

Contributions



and $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$



⇕

SAC

⇓

whether $P(\mathbf{y} \mid do(\mathbf{x})) =$ SCA?

⇕

minSCA

⇓

$\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$

satisfying $P(\mathbf{y} \mid do(\mathbf{x})) =$ SCA